Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol.18, No.1&2, January-June, 2019, 15-24

ON A CERTAIN CLASS OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH FRACTIONAL CALCULUS OPERATOR

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Abstract: The purpose of the present paper is to introduce a new subclass of harmonic univalent functions associated with fractional calculus operator. We obtain coefficient conditions, extreme points, distortion bounds, convolution condition and convex combination for the above class of harmonic univalent functions. Relevant connections of the results presented herewith various well-known results are briefly indicated.

Keywords: Harmonic, Univalent functions, Fractional Calculus.

AMS 2010 Mathematics Subject Classification. 30C45, 30C50.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected complex domain *D* is said to be harmonic in *D* if both *u* and *v* are real harmonic in *D*. In any simply-connected domain we can write $f = h + \overline{g}$, where *h* and *g* are analytic in *D*. We call *h* the analytic part and *g* the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in *D* is that $|h'(z)| > |g'(z)|, z \in D.$ See Clunie and Sheil-Small [2], for more basic results on harmonic functions one may refer to the following standard text book by Duren [6], (see also [1] and [8]) . Further we denote by S_H the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0.$ Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions *h* and *g* as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , \ g(z) = \sum_{k=1}^{\infty} b_k z^k , \ \ |b_1| < 1 . \tag{1}$$

It is easy to verify that the class S_H reduces to class S of normalized analytic univalent functions if co-analytic part of f i.e. $g \equiv 0$, in this class $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2}$$

Further, A denotes the class of functions f of form (2) which are analytic in the open unit disk U.

In 2011, Dixit and Porwal [5] (see also [4]) introduce a new fractional derivative operator by using fractional calculus. First we recall the definitions of fractional derivative operator which are due to Owa [7] and Srivastava and Owa [13].

Definition 1.1. The fractional derivative of order λ is defined, for a function $f(z)$ of the form (2), given by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\varsigma)}{(z-\varsigma)^{\lambda}} d\varsigma,$$

where $0 \le \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin.

Definition 1.2. Under the hypothesis of Definition 1.1, the fractional derivative of order $n+\lambda$ is defined for a function $f(z)$ by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z)$$

where $0 \le \lambda < 1$, and $n \in N_0 = \{0,1,2.....\}$.

For f of the form (2), using the Definition 1.1 and 1.2, Dixit and Porwal [5] introduce a new fractional derivative operator $\Omega^n : A \to A$ in the following way

$$\Omega^0 f(z) = f(z)$$

$$\Omega^1 f(z) = \Gamma(1-\lambda) z^{1+\lambda} D_z^{1+\lambda} f(z)$$

$$\Omega^n f(z) = \Omega\left(\Omega^{n-1} f(z)\right).$$

It is easy to see that

$$\Omega^n f(z) = z + \sum_{k=2}^{\infty} [\phi(k,\lambda)]^n a_k z^k .$$

where

$$\phi(k,\lambda)=\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}.$$

For $\lambda=0$, $\Omega^n f(z)$ reduces to the well-known Salagean operator introduced in [12].

For $f=h+\overline{g}$ given by (1) the operator $\Omega^n f(z)$ is defined as

$$\Omega^n f(z)=\Omega^n h(z)+(-1)^n\overline{\Omega^n g(z)} \tag{3}$$

where

$$\Omega^n h(z)=z+\sum_{k=2}^{\infty}[\phi(k,\lambda)]^n a_k z^k \text{ and } \Omega^n g(z)=\sum_{k=1}^{\infty}[\phi(k,\lambda)]^n b_k z^k .$$

Now for $1<\beta\le 4/3$, $0\le\lambda<1, 0\le t\le 1, m\in N, n\in N_0, m>n$ and $z\in U$, suppose that $S_H^{\lambda}(m,n;\beta;t)$ denote the family of harmonic functions f of the form (1) such that

$$\operatorname{Re}\left\{\frac{\Omega^m f(z)}{\Omega^n f_t(z)}\right\}<\beta, \tag{4}$$

where $f_t(z)=(1-t)z+tf(z)$ and $\Omega^m f$ is defined by (3).

Further let the subclass $VS_H^{\lambda}(m,n;\beta;t)$ consist of harmonic functions $f_m=h+\overline{g_m}$ in $S_H^{\lambda}(m,n;\beta;t)$ so that h and g_m are of the form

$$h(z)=z+\sum_{k=2}^{\infty}|a_k|z^k,\ g_m(z)=(-1)^m\sum_{k=1}^{\infty}|b_k|z^k . \tag{5}$$

Assigning specific values to the parameters in subclass $S_H^{\lambda}(m,n;\beta;t)$ we obtain the following known subclasses studied earlier by various researchers.

1. If we put $\lambda=0$, $m=n+1$ and $t=1$ then it reduces to the class $S_H(n,\beta)$ studied by Porwal and Dixit [11].
2. If we put $t=0$ then it reduces to the class $S_H^{\lambda}(m,\beta)$ studied by Porwal and Aouf [9].
3. If we put $m=1, n=0, \lambda=0, t=1$ and $m=2, n=1, \lambda=0, t=1$ then it reduces to the class $L_H(\beta)$ and $M_H(\beta)$ studied by Porwal and Dixit [10].

4. If we put $m=n+1, \lambda=0, t=1$ with $g\equiv 0$ then it reduces to the class $S(n,\beta)$ studied by Dixit and Pathak [3].
5. If we put $m=1, n=0, \lambda=0, t=1$ and $m=2, n=1, \lambda=0, t=1$ with $g\equiv 0$ then it reduces to the class $L(\beta)$ and $M(\beta)$ studied by Uralegaddi et al. [14].

In the present paper, results involving the coefficient inequalities, extreme points, distortion bounds, convolution condition and convex combinations for the above classes $S_H^\lambda(m,n;\beta;t)$ and $VS_H^\lambda(m,n;\beta;t)$ of harmonic univalent functions have been investigated.

2. Main Results

We begin with a sufficient coefficient condition for functions in $S_H^\lambda(m,n;\beta;t)$.

Theorem 2.1. Let $f=h+\bar{g}$ be such that h and g are given by (1). Furthermore, let

$$\sum_{k=2}^{\infty}\frac{[\varphi(k,\lambda)]^m-\beta t[\varphi(k,\lambda)]^n}{\beta-1}|a_k|+\sum_{k=1}^{\infty}\frac{[\varphi(k,\lambda)]^m-(-1)^{m-n}\beta t[\varphi(k,\lambda)]^n}{\beta-1}|b_k|\le 1, \quad (6)$$

where $m\in N, n\in N_0, m>n, 1<\beta\le 4/3, 0\le\lambda<1, 0\le t\le 1$ and $\phi(k,\lambda)=\frac{\Gamma(k+1)\Gamma(1-\lambda)}{\Gamma(k-\lambda)}$,

then f is sense- preserving, harmonic univalent in U and $f\in S_H^\lambda(m,n;\beta;t)$.

Proof. First we note that f is sense-preserving in U. This is because

$$\begin{aligned}|h'(z)| &\ge 1-\sum_{k=2}^{\infty}k|a_k|r^{k-1}\\ &>1-\sum_{k=2}^{\infty}k|a_k|\\ &\ge 1-\sum_{k=2}^{\infty}\frac{[\varphi(k,\lambda)]^m-\beta t[\varphi(k,\lambda)]^n}{\beta-1}|a_k|\\ &\ge \sum_{k=1}^{\infty}\frac{[\varphi(k,\lambda)]^m-(-1)^{m-n}\beta t[\varphi(k,\lambda)]^n}{\beta-1}|b_k|\\ &\ge \sum_{k=1}^{\infty}k|b_k|\end{aligned}$$

$$> \sum_{k=1}^{\infty} k|b_k| r^{k-1}$$

$$\geq |g'(z)|.$$

To show that f is univalent in *U*, suppose $z_1, z_2 \in U$ such that $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{z_1 - z_2 + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)}\right|$$

$$>1-\frac{\sum_{k=1}^{\infty} k\,|\,b_k\,|}{1-\sum_{k=2}^{\infty} k\,|\,a_k\,|}$$

$$\geq 1-\frac{\sum_{k=1}^{\infty} \frac{\left[\varphi(k,\lambda)\right]^m - (-1)^{m-n}\beta t\left[\varphi(k,\lambda)\right]^n}{\beta-1}|b_k|}{1-\sum_{k=2}^{\infty} \frac{\left[\varphi(k,\lambda)\right]^m - \beta t\left[\varphi(k,\lambda)\right]^n}{\beta-1}|a_k|}$$

$$\geq 0.$$

Using the fact that $\Re\{\omega\} < \beta$, if and only if, $|\,\omega-1| < |\,\omega+1-2\beta\,|$, it suffices to show that

$$\left|\frac{\dfrac{\Omega^m f(z)}{\Omega^n f_t(z)}-1}{\dfrac{\Omega^m f(z)}{\Omega^n f_t(z)}+1-2\beta}\right| < 1, \quad z \in U.$$

Now $\left|\frac{\dfrac{\Omega^m f(z)}{\Omega^n f_t(z)}-1}{\dfrac{\Omega^m f(z)}{\Omega^n f_t(z)}+1-2\beta}\right|$

$$=\left|\frac{\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-t\left[\varphi(k,\lambda)\right]^n\right]a_k z^k+\sum_{k=1}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(-1)^n t\left[\varphi(k,\lambda)\right]^n\right]\overline{b_k z^k}}{2(1-\beta)z+\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]a_k z^k+\sum_{k=1}^{\infty}\left[(-1)^m\left[\varphi(k,\lambda)\right]^m+(-1)^n(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]\overline{b_k z^k}}\right|$$

$$\leq\frac{\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-t\left[\varphi(k,\lambda)\right]^n\right]|a_k||z|^{k-1}+\sum_{k=1}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(-1)^{m-n}t\left[\varphi(k,\lambda)\right]^n\right]|b_k||z|^{k-1}}{2(\beta-1)-\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]|a_k||z|^{k-1}+\sum_{k=1}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m+(-1)^{m-n}(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]|b_k||z|^{k-1}}$$

$$<\frac{\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-t\left[\varphi(k,\lambda)\right]^n\right]|a_k|+\sum_{k=1}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(-1)^{m-n}t\left[\varphi(k,\lambda)\right]^n\right]|b_k|}{2(\beta-1)-\sum_{k=2}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m-(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]|a_k|+\sum_{k=1}^{\infty}\left[\left[\varphi(k,\lambda)\right]^m+(-1)^{m-n}(1-2\beta)t\left[\varphi(k,\lambda)\right]^n\right]|b_k|}$$

≤ 1, (from (6)).

The Coefficient bound given by (6) is sharp for the harmonic univalent functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}\frac{\beta-1}{\left(\left[\varphi(k,\lambda)\right]^m-\beta t\left[\varphi(k,\lambda)\right]^n\right)}x_k z^k+\sum_{k=1}^{\infty}\frac{\beta-1}{\left(\left[\varphi(k,\lambda)\right]^m-(-1)^{m-n}\beta t\left[\varphi(k,\lambda)\right]^n\right)}\overline{y_k z^k}$$

where $1<\beta\leq 4/3$, $0\leq\lambda<1, 0\leq t\leq 1, m\in N, n\in N_0, m>n$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$.

This completes the proof of theorem.

In the following theorem, it is proved that the condition (6) is also necessary for functions $f_m=h+\overline{g_m}$, where h and g_m are of the form (5).

Theorem 2.2. Let $f_m=h+\overline{g_m}$ be given by (5). Then $f_m\in VS_H^{\lambda}(m,n;\beta;t)$, if and only if

$$\sum_{k=2}^{\infty}\left(\left(\varphi(k,\lambda)\right)^m-\beta t\left(\varphi(k,\lambda)\right)^n\right)|a_k|+\sum_{k=1}^{\infty}\left(\left(\varphi(k,\lambda)\right)^m-(-1)^{m-n}\beta t\left(\varphi(k,\lambda)\right)^n\right)|b_k|\leq\beta-1. \quad (7)$$

Proof. Since $VS_H^{\lambda}(m,n;\beta;t)\subset S_H^{\lambda}(m,n;\beta;t)$, we only need to prove the "only if " part of the theorem. To this end, for functions f_m of the form (5), we notice that the condition

$$\operatorname{Re}\left\{\frac{\Omega^m f(z)}{\Omega^n f_t(z)}\right\}<\beta$$

is equivalent to

$$\mathrm{Re}\left\{\frac{(\beta-1)z-\sum_{k=2}^{\infty}\left(\left(\varphi(k,\lambda)\right)^{m}-\beta t\left(\varphi(k,\lambda)\right)^{n}\right)|a_k|z^k-\sum_{k=1}^{\infty}\left(\left(\varphi(k,\lambda)\right)^{m}-(-1)^{m-n}\beta t\left(\varphi(k,\lambda)\right)^{n}\right)|b_k|\bar{z}^k}{z+\sum_{k=2}^{\infty}\left(\varphi(k,\lambda)\right)^{n}t|a_k|z^k+(-1)^{m+n}\sum_{k=1}^{\infty}\left(\varphi(k,\lambda)\right)^{n}t|b_k|\bar{z}^k}\right\}\geq 0. \quad (8)$$

The above required condition (8) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(\beta-1)-\sum_{k=2}^{\infty}\left(\left(\varphi(k,\lambda)\right)^{m}-\beta t\left(\varphi(k,\lambda)\right)^{n}\right)|a_k|r^{k-1}-\sum_{k=1}^{\infty}\left(\left(\varphi(k,\lambda)\right)^{m}-(-1)^{m-n}\beta t\left(\varphi(k,\lambda)\right)^{n}\right)|b_k|r^{k-1}}{1+\sum_{k=2}^{\infty}\left(\varphi(k,\lambda)\right)^{n}t|a_k|r^{k-1}+(-1)^{m+n}\sum_{k=1}^{\infty}\left(\varphi(k,\lambda)\right)^{n}t|b_k|r^{k-1}}\geq 0. \quad (9)$$

If the condition (7) does not hold, then the numerator in (9) is negative for r sufficiently close to 1. Thus there exist a $z_0=r_0$ in (0,1) for which the quotient in (9) is negative. This contradicts the required condition for $f_m\in VS_H^{\lambda}(m,n;\beta;t)$ and so the proof is complete.

The proof of the following Theorems are much akin to the corresponding results of Porwal and Dixit [11], therefore we state the results only.

Theorem 2.3. Let f_m be given by (5). Then $f_m\in VS_H^{\lambda}(m,n;\beta;t)$, if and only if

$$f_m(z)=\sum_{k=1}^{\infty}\left(x_k h_k(z)+y_k g_{mk}(z)\right), \quad (10)$$

where $h_1(z)=z$, $h_k(z)=z+\dfrac{\beta-1}{\left(\varphi(k,\lambda)\right)^{m}-\beta t\left(\varphi(k,\lambda)\right)^{n}}z^k$, $(k=2,3,4....)$,

$$g_{mk}(z)=z+(-1)^m\frac{\beta-1}{\left(\varphi(k,\lambda)\right)^{m}-(-1)^{m-n}\beta t\left(\varphi(k,\lambda)\right)^{n}}\bar{z}^k,$$

$(k=1,2,3....)$, $x_k\geq 0$, $y_k\geq 0$, $\sum_{k=1}^{\infty}(x_k+y_k)=1$. In particular, the extreme points of $VS_H^{\lambda}(m,n;\beta;t)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Theorem 2.4. Let $f_m\in VS_H^{\lambda}(m,n;\beta;t)$. Then for $|z|=r<1$ we have

$$|f_m(z)| \le (1+|b_1|)r + \left(\frac{1-\lambda}{2}\right)^n \left(\frac{\beta-1}{\left(\frac{2}{1-\lambda}\right)^{m-n} - \beta t} - \frac{1-(-1)^{m-n}\beta t}{\left(\frac{2}{1-\lambda}\right)^{m-n} - \beta t}|b_1| \right) r^2,$$

and

$$|f_m(z)| \ge (1-|b_1|)r - \left(\frac{1-\lambda}{2}\right)^n \left(\frac{\beta-1}{\left(\frac{2}{1-\lambda}\right)^{m-n} - \beta t} - \frac{1-(-1)^{m-n}\beta t}{\left(\frac{2}{1-\lambda}\right)^{m-n} - \beta t}|b_1| \right) r^2.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary. Let f_m of the form (5) be so that $f_m \in VS_H^{\lambda}(m,n;\beta;t)$.Then

$$\left\{ \omega : |\omega| < \frac{(2/1-\lambda)^m - \beta t(2/1-\lambda)^n - (\beta-1)}{(2/1-\lambda)^m - \beta t(2/1-\lambda)^n} - \frac{(2/1-\lambda)^m - \beta t(2/1-\lambda)^n - \left(1-(-1)^{m-n}\beta t\right)}{(2/1-\lambda)^m - \beta t(2/1-\lambda)^n}|b_1| \right\} \subset f_m(U).$$

Theorem 2.5. For $1 < \beta \le \alpha \le \frac{4}{3}$, let $f_m(z) \in VS_H^{\lambda}(m,n;\beta;t)$ and $F_m(z) \in VS_H^{\lambda}(m,n;\alpha;t)$.

Then $f_m * F_m \in \overline{S}_H^{\lambda}(m,n;\alpha;t) \subset \overline{S}_H^{\lambda}(m,n;\beta;t)$.

Theorem 2.6. The class $VS_H^{\lambda}(m,n;\beta;t)$ is closed under convex combination.

Acknowledgements

The authors are thankful to the Referee for valuable comments and suggestions.

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