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DUALLY FLAT FINSLER SPACES ASSOCIATED WITH RANDERS-β CHANGE

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Abstract: The present paper deals with the property of dually flatness of Finsler spaces with some special $(\alpha, \beta)-$ metrics constructed in [10] via Randers-β change. Here, we find necessary and sufficient conditions under which the Randers change of these $(\alpha, \beta)-$ metrics are locally dually flat. Finally, we conclude the interrelation between the locally dually flatness of all these metrics.

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1. Introduction

Finsler geometry is just Riemannian geometry without quadratic restriction as described by Chern [5]. Finsler geometry is an interesting and active area of research for both applied and pure mathematicians [2]. Since, the emergence of this geometry, though there has been a lot of development, still there is a huge scope of research work. The idea of (α, β) $-$metric in Finsler geometry, was first introduced by Matsumoto in [7]. In 1984, Shibata [13] introduced the notion of β $-$change in Finsler geometry. The concept of dually flatness in Riemannian geometry was given by Amari and Nagaoka in [1] while studying information geometry. Information geometry provides mathematical science with a new framework for analysis. Information geometry is an investigation of differential geometric structure in probability distribution. It is also applicable in statistical physics, statistical inferences etc. Shen [12] extended the notion of dually flatness in Finsler spaces. After Shen's work, many authors have worked on this topic (see [3, 9, 11]).

The current paper is organized as follows:

In section two, we give basic definitions and examples of some special Finsler spaces with (α, β) $-$metrics obtained by Randers change. In sections three, we find necessary and sufficient conditions for Randers change of some special Finsler spaces with (α,β) −metrics to be locally dually flat.

2. Preliminaries

The literature of Riemann-Finsler geometry has been developed rapidly by so many geometers across the Globe during last few decades. Here, we discuss some basic definitions, examples and results which are required for subsequent sections. For symbols and notations, we refer [4, 6].

Definition 2.1 *An n-dimensional real vector space V is called a Minkowski space if there exists a real valued function $F: V \longrightarrow [0,\infty)$, called Minkowski norm, satisfying the following conditions:*

1. F is smooth on $V\backslash\{0\}$,
2. F is positively homogeneous, i.e., $F(\lambda v)=\lambda F(v), \forall\, \lambda>0$,
3. For a basis $\{v_1, v_2, \ldots, v_n\}$ of V and $y=y^i v_i \in V$, the Hessian matrix $\left(g_{ij}\right)=\left(\frac{1}{2}F^2_{y^i y^j}\right)$ is positive-definite at every point of $V\backslash\{0\}$.

Definition 2.2 *A connected smooth manifold M is called a Finsler space if there exists a function $F: TM \longrightarrow [0,\infty)$ such that F is smooth on the slit tangent bundle $TM\backslash\{0\}$ and the restriction of F to any $T_pM, p\in M$, is a Minkowski norm. Here, F is called a Finsler metric.*

The notion of (α,β)-metrics was introduced by Matsumoto in [7]. Basically, an (α,β)-metric on a connected smooth manifold M is a Finsler metric F constructed from a Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^i y^j}$ and a one-form $\beta=b_i(x)y^i$ on M and is of the form $F=\alpha\phi\left(\frac{\beta}{\alpha}\right)$, where ϕ is a smooth function on M. Basically, (α,β)-metrics are the generalization of Randers metrics. Let us recall Shen's lemma [4, 6] which provides necessary and sufficient condition for an (α,β)-metric to be a Finsler metric.

Lemma 2.1 *Let $F=\alpha\phi(s), s=\beta/\alpha$, where ϕ is a smooth function on $(-b_0,b_0)$, α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < b_0$. Then F is a Finsler metric if and only if the following conditions are satisfied:*

$\phi(s)>0, \phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0, \forall\, |s|\le b<b_0$.

Some of the classical examples of (α,β) −metrics are as follows:

Randers metric, Kropina metric, generalized Kropina metric, Z. Shen's square metric, Matsumoto metric, exponential metric, infinite series metric.

Next, we recall [13] the following:

Definition 2.3 *Let (M,F) be an n −dimensional Finsler space. Then a Finsler metric $\bar{F}=f(F,\beta)+\beta$, constructed via β −change is known as Randers change of (α,β) −metric.*

Following are some special Finsler metrics constructed by Shanker et al. (see [10]) via Randers change of (α, β) metrics. Our further studies will be based on these metrics.

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a $1-$form. Then, we construct the following:

1. Matsumoto-Randers changed (α, β)-metric:

We know that $F = \frac{\alpha^2}{\alpha-\beta}$ is Matsumoto metric. Applying $\beta-$change and Randers change simultaneously to this metric, we obtain a new metric $\bar{F} = \frac{F^2}{F-\beta} + \beta$, which we call Matsumoto-Randers changed metric.

2. Exponential-Randers Changed (α, β) Metric

The metric $F = \alpha e^{\beta/\alpha}$ is called exponential metric. Applying $\beta-$change and Randers change simultaneously to this metric, we obtain a new metric $\bar{F} = Fe^{\beta/F} + \beta$, which we call exponential-Randers changed metric.

3. Infinite Series-Randers changed (α, β)-Metric

The metric $F = \frac{\beta^2}{\beta-\alpha}$ is known as infinite series metric. Applying $\beta-$change and Randers change simultaneously to this metric, we obtain a new metric $\bar{F} = \frac{\beta^2}{\beta-F} + \beta$, which we call infinite series-Randers changed metric.

Recall [8, 14] the following definition:

Definition 2.4 *Let (M, F) be an $n-$dimensional Finsler space. If $F = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}}$, with $A := a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}$ symmetric in all the indices, then F is called m^{th}-root Finsler metric.*

Remark 1 *In the original definition, it was written as $L = \sqrt[m]{a_{i_1 i_2 \dots i_m} y^{i_1} y^{i_2} \dots y^{i_m}}$, but we are using F to be (α, β)-metric. That's why we have written in this way.*

m^{th}-root Finsler metric has applications in Ecology [1]. This metric is generalization of Riemannian metric as the square root metric is a Riemannian metric. We will focus on Randers-β change of square root Finsler metrics in this paper. We use following notations in the subsequent sections:

$$\frac{\partial L}{\partial x^i} = L_{x^i}, \frac{\partial L}{\partial y^i} = L_{y^i}, \frac{\partial A}{\partial x^i} = A_{x^i}, \frac{\partial A}{\partial y^i} = A_i, A_{x^i} y^i = A_0, A_{x^i y^j} y^i = A_{0j},$$
$$\frac{\partial \beta}{\partial x^i} = \beta_{x^i}, \frac{\partial \beta}{\partial y^i} = b_i \text{ or } \beta_i, \beta_{x^i} y^i = \beta_0, \beta_{x^i y^j} y^i = \beta_{0j} \text{ etc.}$$

3. Dually Flatness of Finsler Metrics

Firstly, we recall [12] the following definition:

Definition 3.1 *A Finsler metric F on a smooth n −manifold M is called locally dually flat if, at any point, there is a standard co-ordinate system (x^i, y^i) in TM, (x^i) called adapted local co-ordinate system, such that*

$L_{x^k y^m} y^k - 2L_{x^m} = 0,$ where $L = F^2$.

Next, we find the necessary and sufficient conditions for locally dually flatness of all the metrics constructed via Randers-β change in section 2.

First, we find necessary and sufficient conditions for Matsumoto-Randers changed Finsler metric

$\bar{F} = \frac{F^2}{F-\beta} + \beta$ to be locally dually flat.

Let us put $F^2 = A$ in $\bar{F}$, then $\bar{F} = \frac{A}{\sqrt{A}-\beta} + \beta$,

which implies

$\bar{L} = \bar{F}^2 = \frac{A^2}{(\sqrt{A}-\beta)^2} + \frac{2A\beta}{\sqrt{A}-\beta} + \beta^2.(1)$

Differentiating (1) w.r.t. x^k, we get

$$\bar{L}_{x^k} = \frac{2A}{(\sqrt{A}-\beta)^2} A_{x^k} - \frac{A^{\frac{3}{2}}}{(\sqrt{A}-\beta)^3} A_{x^k} + \frac{2A^2}{(\sqrt{A}-\beta)^3} \beta_{x^k} + \frac{2\beta}{\sqrt{A}-\beta} A_{x^k} + \frac{2A}{\sqrt{A}-\beta} \beta_{x^k}$$

$$- \frac{\sqrt{A}\beta}{(\sqrt{A}-\beta)^2} A_{x^k} + \frac{2A\beta}{(\sqrt{A}-\beta)^2} \beta_{x^k} + 2\beta\beta_{x^k}. \qquad (2)$$

Differentiation of (2) further w.r.t. y^ℓ gives

$$\bar{L}_{x^k y^\ell} = \frac{2A}{(\sqrt{A}-\beta)^2} A_{x^k y^\ell} + \frac{2}{(\sqrt{A}-\beta)^2} A_{x^k} A_\ell - \frac{4A}{(\sqrt{A}-\beta)^3} A_{x^k} \left(\frac{1}{2\sqrt{A}} A_\ell - \beta_\ell\right) - \frac{A^{3/2}}{(\sqrt{A}-\beta)^3} A_{x^k y^\ell}$$

$$- \frac{3A^{1/2}}{2(\sqrt{A}-\beta)^3} A_{x^k} A_\ell + \frac{3A^{3/2}}{(\sqrt{A}-\beta)^4} A_{x^k} \left(\frac{1}{2\sqrt{A}} A_\ell - \beta_\ell\right) + \frac{2A^2}{(\sqrt{A}-\beta)^3} \beta_{x^k y^\ell} + \frac{4A}{(\sqrt{A}-\beta)^3} \beta_{x^k} A_\ell$$

$$- \frac{6A^2}{(\sqrt{A}-\beta)^4} \beta_{x^k} \left(\frac{1}{2\sqrt{A}} A_\ell - \beta_\ell\right) + \frac{2\beta}{\sqrt{A}-\beta} A_{x^k y^\ell} + \frac{2}{\sqrt{A}-\beta} A_{x^k} \beta_\ell - \frac{2\beta}{(\sqrt{A}-\beta)^2} A_{x^k} \left(\frac{1}{2\sqrt{A}} A_\ell - \beta_\ell\right)$$

$$+ \frac{2A}{\sqrt{A}-\beta} \beta_{x^k y^\ell} + \frac{2}{\sqrt{A}-\beta} \beta_{x^k} A_\ell - \frac{2A}{(\sqrt{A}-\beta)^2} \beta_{x^k} \left(\frac{1}{2\sqrt{A}} A_\ell - \beta_\ell\right) - \frac{\sqrt{A}\beta}{(\sqrt{A}-\beta)^2} A_{x^k y^\ell}$$

$$-\frac{\sqrt{A}}{(\sqrt{A}-\beta)^2}A_{x^k}\beta_\ell-\frac{\beta}{2\sqrt{A}(\sqrt{A}-\beta)^2}A_{x^k}A_\ell+\frac{2\sqrt{A}\beta}{(\sqrt{A}-\beta)^3}A_{x^k}\left(\frac{1}{2\sqrt{A}}A_\ell-\beta_\ell\right)+\frac{2A\beta}{(\sqrt{A}-\beta)^2}\beta_{x^k y^\ell}$$
$$+\frac{2A}{(\sqrt{A}-\beta)^2}\beta_{x^k}\beta_\ell+\frac{2\beta}{(\sqrt{A}-\beta)^2}\beta_{x^k}A_\ell-\frac{4A\beta}{(\sqrt{A}-\beta)^3}\beta_{x^k}\left(\frac{1}{2\sqrt{A}}A_\ell-\beta_\ell\right)+2\beta\beta_{x^k y^\ell}+2\beta_{x^k}\beta_\ell. \tag{3}$$

Contracting (3) with y^k and simplifying, we get

$$\begin{aligned}\bar{L}_{x^k y^\ell}y^k = \frac{1}{2\sqrt{A}(\sqrt{A}-\beta)^4}&[8A^3\beta_{0\ell}+2A^{5/2}\{(A_{0\ell}+10\beta_0\beta_\ell+2\beta_{0\ell})-4\beta\beta_{0\ell}\}\\ &+4A^2\{(A_0\beta_\ell+\beta_0A_\ell)\\ &-\beta(A_{0\ell}+4\beta_0\beta_\ell+2\beta_{0\ell})-3\beta^2\beta_{0\ell}\}-4A^{\frac{3}{2}}\{4\beta(A_0\beta_\ell+\beta_0A_\ell)\\ &\beta^2(A_{0\ell}-5\beta_0\beta_\ell-\beta_{0\ell})-6\beta^3\beta_{0\ell}\}\\ &-2A\{\beta A_0A_\ell-3\beta^2(A_0\beta_\ell+\beta_0A_\ell)-\beta^3(5A_{0\ell}-8\beta_0\beta_\ell)+8\beta^4\beta_{0\ell}\}\\ &+4A^{1/2}\{2\beta^2A_0A_\ell+\beta^4(\beta_0\beta_\ell-A_{0\ell})+\beta^5\beta_{0\ell}\}-3\beta^3A_0A_\ell].\end{aligned}$$

Further, equation (2) can be rewritten as

$$\begin{aligned}2\bar{L}_{x^\ell}=\frac{1}{2\sqrt{A}(\sqrt{A}-\beta)^4}&[16A^3\beta_{x^\ell}+2A^{5/2}\{2A_{x^\ell}-8\beta\beta_{x^\ell}\}-4A^2\{2\beta A_{x^\ell}+6\beta^2\beta_{x^\ell}\}\\ &-4A^{3/2}\{2\beta^2A_{x^\ell}-12\beta^3\beta_{x^\ell}\}+2A\{10\beta^3A_{x^\ell}-16\beta^4\beta_{x^\ell}\}-4A^{1/2}\{2\beta^4A_{x^\ell}-2\beta^5\beta_{x^\ell}\}].\end{aligned}$$

We know that $\bar{F}$ is locally dually flat if and only if $\bar{L}_{x^k y^\ell}y^k-2\bar{L}_{x^\ell}=0$, i.e.,

$$\begin{aligned}&8A^3\{\beta_{0\ell}-2\beta_{x^\ell}\}+2A^{5/2}\{(A_{0\ell}-2A_{x^\ell}+10\beta_0\beta_\ell+2\beta_{0\ell})-4\beta(\beta_{0\ell}-2\beta_{x^\ell})\}\\ &+4A^2\{(A_0\beta_\ell+\beta_0A_\ell)-\beta(A_{0\ell}-2A_{x^\ell}+4\beta_0\beta_\ell+2\beta_{0\ell})-3\beta^2(\beta_{0\ell}-2\beta_{x^\ell})\}\\ &-4A^{3/2}\{4\beta(A_0\beta_\ell+\beta_0A_\ell)+\beta^2(A_{0\ell}-2A_{x^\ell}-5\beta_0\beta_\ell-\beta_{0\ell})-6\beta^3(\beta_{0\ell}-2\beta_{x^\ell})\}\\ &-2A\{\beta A_0A_\ell-3\beta^2(A_0\beta_\ell+\beta_0A_\ell)-\beta^3(5A_{0\ell}-10A_{x^\ell}-8\beta_0\beta_\ell)+8\beta^4(\beta_{0\ell}-2\beta_{x^\ell})\}\\ &+4A^{1/2}\{2\beta^2A_0A_\ell+\beta^4(\beta_0\beta_\ell-A_{0\ell}+2A_{x^\ell})+\beta^5(\beta_{0\ell}-2\beta_{x^\ell})\}-3\beta^3A_0A_\ell=0.\end{aligned}$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following seven equations are satisfied.

$$\beta_{0\ell}=2\beta_{x^\ell} \tag{4}$$

$$(A_{0\ell}-2A_{x^\ell}+10\beta_0\beta_\ell+2\beta_{0\ell})-4\beta(\beta_{0\ell}-2\beta_{x^\ell})=0 \tag{5}$$

$$(A_0\beta_\ell+\beta_0A_\ell)-\beta(A_{0\ell}-2A_{x^\ell}+4\beta_0\beta_\ell+2\beta_{0\ell})-3\beta^2(\beta_{0\ell}-2\beta_{x^\ell})=0 \tag{6}$$

$$4\beta(A_0\beta_\ell+\beta_0A_\ell)+\beta^2(A_{0\ell}-2A_{x^\ell}-5\beta_0\beta_\ell-\beta_{0\ell})-6\beta^3(\beta_{0\ell}-2\beta_{x^\ell})=0 \tag{7}$$

$$\beta A_0 A_\ell - 3\beta^2(A_0\beta_\ell + \beta_0 A_\ell) - \beta^3(5A_{0\ell} - 10A_{x^\ell} - 8\beta_0\beta_\ell) + 8\beta^4(\beta_{0\ell} - 2\beta_{x^\ell}) = 0 \quad (8)$$

$$2\beta^2 A_0 A_\ell + \beta^4(\beta_0\beta_\ell - A_{0\ell} + 2A_{x^\ell}) + \beta^5(\beta_{0\ell} - 2\beta_{x^\ell}) = 0 \quad (9)$$

$$A_0 A_\ell = 0. \quad (10)$$

Solving above seven equations, we get

$$A_0 A_\ell = 0, A_{0\ell} = 2A_{x^\ell}, \beta_0\beta_\ell = 0, \beta_{0\ell} = 0 = 2\beta_{x^\ell}, A_\ell\beta_0 + A_0\beta_\ell = 0.$$

Above discussion leads to the following theorem.

Theorem 3.1 *Let* $(M, \bar{F})$ *be an* $n-$*dimensional Finsler space with* $\bar{F} = \frac{F^2}{F-\beta} + \beta$ *as a Matsumoto-Randers changed metric. Then* $\bar{F}$ *is locally dually flat if and only if the following equations are satisfied:*

$$A_0 A_m = 0, A_{0m} = 2A_{x^m}, \beta_0\beta_m = 0, \beta_{0m} = 0 = 2\beta_{x^m}, A_m\beta_0 + A_0\beta_m = 0.$$

Next, we find necessary and sufficient conditions for exponential-Randers changed Finsler metric

$\bar{F} = Fe^{\beta/F} + \beta$ to be locally dually flat.

Let us put $F^2 = A$ in $\bar{F}$, then $\bar{F} = \sqrt{A}e^{\beta/\sqrt{A}} + \beta$,

which implies

$$\bar{L} = \bar{F}^2 = Ae^{2\beta/\sqrt{A}} + 2\sqrt{A}\beta e^{\beta/\sqrt{A}} + \beta^2. \quad (11)$$

Differentiating (11) w.r.t. x^k, we get

$$\begin{aligned}\bar{L}_{x^k} = \ & 2\sqrt{A}e^{2\beta/\sqrt{A}}\beta_{x^k} - \frac{\beta}{\sqrt{A}}e^{2\beta/\sqrt{A}}A_{x^k} + e^{2\beta/\sqrt{A}}A_{x^k} + \frac{\beta}{\sqrt{A}}e^{\beta/\sqrt{A}}A_{x^k} \\ & + 2\sqrt{A}e^{\beta/\sqrt{A}}\beta_{x^k} + 2\beta e^{\beta/\sqrt{A}}\beta_{x^k} - \frac{\beta^2}{A}e^{\beta/\sqrt{A}}A_{x^k} + 2\beta\beta_{x^k}.\end{aligned} \quad (12)$$

Differentiation of (12) further w.r.t. y^m gives

$$\begin{aligned}\bar{L}_{x^k y^m} = \ & 2\sqrt{A}e^{2\beta/\sqrt{A}}\beta_{x^k y^m} + \frac{1}{\sqrt{A}}e^{2\beta/\sqrt{A}}\beta_{x^k}A_m + 2\sqrt{A}e^{2\beta/\sqrt{A}}\beta_{x^k}\left(\frac{2}{\sqrt{A}}\beta_m - \frac{\beta}{A^{3/2}}A_m\right) - \frac{\beta}{\sqrt{A}}e^{2\beta/\sqrt{A}}A_{x^k y^m} \\ & - \frac{1}{\sqrt{A}}e^{2\beta/\sqrt{A}}A_{x^k}\beta_m + \frac{\beta}{2A^{3/2}}e^{2\beta/\sqrt{A}}A_{x^k}A_m - \frac{\beta}{\sqrt{A}}e^{2\beta/\sqrt{A}}A_{x^k}\left(\frac{2}{\sqrt{A}}\beta_m - \frac{\beta}{A^{3/2}}A_m\right) + e^{2\beta/\sqrt{A}}A_{x^k y^m} \\ & + e^{2\beta/\sqrt{A}}A_{x^k}\left(\frac{2}{\sqrt{A}}\beta_m - \frac{\beta}{A^{3/2}}A_m\right) + \frac{\beta}{\sqrt{A}}e^{\beta/\sqrt{A}}A_{x^k y^m} + \frac{1}{\sqrt{A}}e^{\beta/\sqrt{A}}A_{x^k}\beta_m - \frac{\beta}{2A^{3/2}}e^{\beta/\sqrt{A}}A_{x^k}A_m \\ & + \frac{\beta}{\sqrt{A}}e^{\beta/\sqrt{A}}A_{x^k}\left(\frac{1}{\sqrt{A}}\beta_m - \frac{\beta}{2A^{3/2}}A_m\right) + 2\sqrt{A}e^{\beta/\sqrt{A}}\beta_{x^k y^m} + \frac{1}{\sqrt{A}}e^{\beta/\sqrt{A}}\beta_{x^k}A_m + 2\sqrt{A}e^{\beta/\sqrt{A}} \\ & \beta_{x^k}\left(\frac{1}{\sqrt{A}}\beta_m - \frac{\beta}{2A^{3/2}}A_m\right) + 2\beta e^{\beta/\sqrt{A}}\beta_{x^k y^m} + 2e^{\beta/\sqrt{A}}\beta_{x^k}\beta_m + 2\beta e^{\beta/\sqrt{A}}\beta_{x^k}\left(\frac{1}{\sqrt{A}}\beta_m - \frac{\beta}{2A^{3/2}}A_m\right) \\ - \frac{\beta^2}{A}e^{\beta/\sqrt{A}} \ & A_{x^k y^m} - \frac{2\beta}{A}e^{\beta/\sqrt{A}}A_{x^k}\beta_m + \frac{\beta^2}{A^2}e^{\beta/\sqrt{A}}A_{x^k}A_m - \frac{\beta^2}{A}e^{\beta/\sqrt{A}}A_{x^k}\left(\frac{1}{\sqrt{A}}\beta_m - \frac{\beta}{2A^{3/2}}A_m\right) + 2\beta\beta_{x^k y^m} + 2\beta_{x^k}\beta_m.\end{aligned} \quad (13)$$

Contracting (13) with y^k and simplifying, we get

$$\bar{L}_{x^k y^m}y^k = \frac{1}{2A^{5/2}}[4A^3 e^{\beta/\sqrt{A}}(e^{\beta/\sqrt{A}} + 1)\beta_{0m} + 2A^{5/2}\{2\beta(e^{\beta/\sqrt{A}}\beta_{0m} + \beta_{0m}) + e^{2\beta/\sqrt{A}}A_{0m}$$

$$+4e^{\beta/\sqrt{A}}(e^{\beta/\sqrt{A}}+1)\beta_0\beta_m+2\beta_0\beta_m\}+2e^{\beta/\sqrt{A}}A^2\{\beta\left((1-e^{\beta/\sqrt{A}})A_{0m}+2\beta_0\beta_m\right)$$
$$+\left(e^{\beta/\sqrt{A}}+1\right)(A_0\beta_m+\beta_0A_m)\}-2e^{\beta/\sqrt{A}}A^{3/2}\{\beta^2A_{0m}+\beta\left(2e^{\beta/\sqrt{A}}+1\right)(A_0\beta_m+\beta_0A_m)\}$$
$$-e^{\frac{\beta}{\sqrt{A}}}A\left\{2(A_0\beta_m+\beta_0A_m)+\beta\left(e^{\frac{\beta}{\sqrt{A}}}+1\right)A_0A_m\right\}+\beta^2e^{\frac{\beta}{\sqrt{A}}}\left(2e^{\frac{\beta}{\sqrt{A}}}+1\right)\sqrt{A}A_0A_m+\beta^3e^{\beta/\sqrt{A}}A_0A_m].$$

Further, equation (12) can be rewritten as

$$2\bar{L}_{x^m}=\frac{1}{2A^{5/2}}[8A^3e^{\beta/\sqrt{A}}\left(e^{\beta/\sqrt{A}}+1\right)\beta_{x^m}-4\beta A^2e^{\beta/\sqrt{A}}\left(e^{\beta/\sqrt{A}}-1\right)A_{x^m}$$
$$+4A^{5/2}\{e^{2\beta/\sqrt{A}}A_{x^m}+2\beta\left(e^{\beta/\sqrt{A}}+1\right)\beta_{x^m}\}-4\beta^2A^{3/2}e^{\beta/\sqrt{A}}A_{x^m}].$$

We know that $\bar{F}$ is locally dually flat if and only if $\bar{L}_{x^ky^m}y^k-2\bar{L}_{x^m}=0$, i.e.,

$$4A^3e^{\beta/\sqrt{A}}\left(e^{\beta/\sqrt{A}}+1\right)\{\beta_{0m}-2\beta_{x^m}\}+2A^{5/2}\{2\beta\left(e^{\beta/\sqrt{A}}+1\right)(\beta_{0m}-2\beta_{x^m})$$
$$+e^{2\beta/\sqrt{A}}(A_{0m}-2A_{x^m})+4e^{\beta/\sqrt{A}}\left(e^{\beta/\sqrt{A}}+1\right)\beta_0\beta_m+2\beta_0\beta_m\}$$
$$+2e^{\beta/\sqrt{A}}A^2\{\beta\left((1-e^{\beta/\sqrt{A}})(A_{0m}-2A_{x^m})+2\beta_0\beta_m\right)+(e^{\beta/\sqrt{A}}+1)(A_0\beta_m+\beta_0A_m)\}$$
$$-2e^{\beta/\sqrt{A}}A^{3/2}\{\beta^2(A_{0m}-2A_{x^m})+\beta\left(2e^{\beta/\sqrt{A}}+1\right)(A_0\beta_m+\beta_0A_m)\}$$
$$-e^{\beta/\sqrt{A}}A\{2(A_0\beta_m+\beta_0A_m)+\beta\left(e^{\beta/\sqrt{A}}+1\right)A_0A_m\}+\beta^2e^{\beta/\sqrt{A}}\left(2e^{\beta/\sqrt{A}}+1\right)\sqrt{A}A_0A_m+\beta^3e^{\beta/\sqrt{A}}A_0A_m=0.$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following seven equations are satisfied.

$$\left(e^{\beta/\sqrt{A}}+1\right)\{\beta_{0m}-2\beta_{x^m}\}=0\beta_{0m}=2\beta_{x^m} \tag{14}$$

$$2\beta\left(e^{\beta/\sqrt{A}}+1\right)(\beta_{0m}-2\beta_{x^m})+e^{2\beta/\sqrt{A}}(A_{0m}-2A_{x^m})+4e^{\beta/\sqrt{A}}\left(e^{\beta/\sqrt{A}}+1\right)\beta_0\beta_m+2\beta_0\beta_m=0 \tag{15}$$

$$\beta\left(\left(1-e^{\beta/\sqrt{A}}\right)(A_{0m}-2A_{x^m})+2\beta_0\beta_m\right)+\left(e^{\beta/\sqrt{A}}+1\right)(A_0\beta_m+\beta_0A_m)=0 \tag{16}$$

$$\beta^2(A_{0m}-2A_{x^m})+\beta\left(2e^{\beta/\sqrt{A}}+1\right)(A_0\beta_m+\beta_0A_m)=0 \tag{17}$$

$$2(A_0\beta_m+\beta_0A_m)+\beta\left(e^{\beta/\sqrt{A}}+1\right)A_0A_m=0 \tag{18}$$

$$\left(2e^{\beta/\sqrt{A}}+1\right)A_0A_m=0A_0A_m=0. \tag{19}$$

Further, from the equations (18) and (19), we get

$$A_0\beta_m + \beta_0 A_m = 0 \tag{20}$$

Again, from the equations (17) and (20), we get

$$A_{0m} = 2A_{x^m}, \tag{21}$$

and from the equations (16), (20) and (21), we get

$$\beta_0\beta_m = 0. \tag{22}$$

Next, we find necessary and sufficient conditions for infinite series-Randers changed Finsler metric

$\bar{F} = \frac{\beta^2}{\beta - F} + \beta$ to be locally dually flat.

Let us put $F^2 = A$ in $\bar{F}$, then

$$\bar{F} = \frac{\beta^2}{\beta - \sqrt{A}} + \beta,$$

which implies

$$\bar{L} = \bar{F}^2 = \frac{\beta^4}{(\beta-\sqrt{A})^2} + \frac{2\beta^3}{\beta-\sqrt{A}} + \beta^2. \tag{23}$$

Differentiating (23) w.r.t. x^k, we get

$$\bar{L}_{x^k} = -\frac{2\beta^3\sqrt{A}}{(\beta-\sqrt{A})^3}\beta_{x^k} + \frac{6\beta^2}{\beta-\sqrt{A}}\beta_{x^k} + 2\beta\beta_{x^k} + \frac{\beta^4}{\sqrt{A}(\beta-\sqrt{A})^3}A_{x^k} + \frac{\beta^3}{\sqrt{A}(\beta-\sqrt{A})^2}A_{x^k}. \tag{24}$$

Differentiation of (24) further w.r.t. y^m gives

$$\begin{aligned}\bar{L}_{x^k y^m} = &-\frac{2\beta^3\sqrt{A}}{(\beta-\sqrt{A})^3}\beta_{x^k y^m} - \frac{6\beta^2\sqrt{A}}{(\beta-\sqrt{A})^3}\beta_{x^k}\beta_m - \frac{\beta^3}{\sqrt{A}(\beta-\sqrt{A})^3}\beta_{x^k}A_m + \frac{6\beta^3\sqrt{A}}{(\beta-\sqrt{A})^4}\beta_{x^k}\left(\beta_m - \frac{A_m}{2\sqrt{A}}\right)\\ &+ \frac{6\beta^2}{\beta-\sqrt{A}}\beta_{x^k y^m} + \frac{12\beta}{\beta-\sqrt{A}}\beta_{x^k}\beta_m - \frac{6\beta^2}{(\beta-\sqrt{A})^2}\beta_{x^k}\left(\beta_m - \frac{1}{2\sqrt{A}}A_m\right) + 2\beta\beta_{x^k y^m} + 2\beta_{x^k}\beta_m\\ &+ \frac{\beta^4}{\sqrt{A}(\beta-\sqrt{A})^3}A_{x^k y^m} + \frac{4\beta^3}{\sqrt{A}(\beta-\sqrt{A})^3}A_{x^k}\beta_m - \frac{\beta^4}{2A^{3/2}(\beta-\sqrt{A})^3}A_{x^k}A_m\\ &- \frac{3\beta^4}{\sqrt{A}(\beta-\sqrt{A})^4}A_{x^k}\left(\beta_m - \frac{1}{2\sqrt{A}}A_m\right) + \frac{\beta^3}{\sqrt{A}(\beta-\sqrt{A})^2}A_{x^k y^m}\\ &+ \frac{3\beta^2}{\sqrt{A}(\beta-\sqrt{A})^2}A_{x^k}\beta_m - \frac{\beta^3}{2A^{3/2}(\beta-\sqrt{A})^2}A_{x^k}A_m - \frac{2\beta^3}{\sqrt{A}(\beta-\sqrt{A})^3}A_{x^k}\left(\beta_m - \frac{1}{2\sqrt{A}}A_m\right).\end{aligned} \tag{25}$$

Contracting (25) with y^k and simplifying, we get

$$\begin{aligned}\bar{L}_{x^k y^m}y^k = &\frac{1}{2A^{3/2}(\beta-\sqrt{A})^4}[4A^{7/2}\{\beta\beta_{0m} + 2\beta_0\beta_m\} - 28A^3\{\beta^2\beta_{0m} + 2\beta\beta_0\beta_m\}\\ &+ 8A^{5/2}\{8\beta^3\beta_{0m} + 15\beta^2\beta_0\beta_m\}\end{aligned}$$

$$+2A^2\{-28\beta^4\beta_{0m} - 40\beta^3\beta_0\beta_m + \beta^3 A_{0m} + 3\beta^2(A_0\beta_m + \beta_0 A_m)\}$$
$$+2A^{3/2}\{8\beta^5\beta_{0m} + 10\beta^4\beta_0\beta_m - 3\beta^4 A_{0m} - 8\beta^3(A_0\beta_m + \beta_0 A_m)\}$$
$$+A\{4\beta^5 A_{0m} + 4\beta^4(A_0\beta_m + \beta_0 A_m) - 3\beta^3 A_0 A_m\} + 8\beta^4\sqrt{A}A_0 A_m - 2\beta^5 A_0 A_m].$$

Further, equation (24) can be rewritten as

$$\bar{L}_{x^m} = \frac{1}{2A^{3/2}(\beta-\sqrt{A})^4}[8\beta A^{7/2}\beta_{x^m} - 56A^3\beta^2\beta_{x^m} + 128A^{5/2}\beta^3\beta_{x^m} - 2A^2\{56\beta^4\beta_{x^m} - 2\beta^3 A_{x^m} + 2A^{3/2}\{16\beta^5\beta_{x^m} - 6\beta^4 A_{x^m}\} + 8\beta^5 A A_{x^m}].$$

We know that $\bar{F}$ is locally dually flat if and only if $\bar{L}_{x^k y^m} y^k - 2\bar{L}_{x^m} = 0$, i.e.,

$$4A^{7/2}\{\beta(\beta_{0m} - 2\beta_{x^m}) + 2\beta_0\beta_m\} - 28A^3\beta\{\beta(\beta_{0m} - 2\beta_{x^m}) + 2\beta_0\beta_m\} + 2A^2\beta^2\{-28\beta^2(\beta_{0m} - 2\beta_{x^m})$$
$$-40\beta\beta_0\beta_m + \beta(A_{0m} - 2A_{x^m}) + 3(A_0\beta_m + \beta_0 A_m)\} + 8A^{5/2}\beta^2\{8\beta(\beta_{0m} - 2\beta_{x^m}) + 15\beta_0\beta_m\}$$
$$+2A^{3/2}\beta^3\{8\beta^2(\beta_{0m} - 2\beta_{x^m}) + 10\beta\beta_0\beta_m - 3\beta(A_{0m} - 2A_{x^m}) - 8(A_0\beta_m + \beta_0 A_m)\}$$
$$+A\beta^3\{4\beta^2(A_{0m} - 2A_{x^m}) + 4\beta(A_0\beta_m + \beta_0 A_m) - 3A_0 A_m\} + 8\beta^4\sqrt{A}A_0 A_m - 2\beta^5 A_0 A_m = 0.$$

From the above equation, we conclude that $\bar{F}$ is locally dually flat if and only if following six equations are satisfied:

$$\beta(\beta_{0m} - 2\beta_{x^m}) + 2\beta_0\beta_m = 0 \quad (26)$$

$$-28\beta^2(\beta_{0m} - 2\beta_{x^m}) - 40\beta\beta_0\beta_m + \beta(A_{0m} - 2A_{x^m}) + 3(A_0\beta_m + \beta_0 A_m) = 0 \quad (27)$$

$$8\beta(\beta_{0m} - 2\beta_{x^m}) + 15\beta_0\beta_m = 0 \quad (28)$$

$$8\beta^2(\beta_{0m} - 2\beta_{x^m}) + 10\beta\beta_0\beta_m - 3\beta(A_{0m} - 2A_{x^m}) - 8(A_0\beta_m + \beta_0 A_m) = 0 \quad (29)$$

$$4\beta^2(A_{0m} - 2A_{x^m}) + 4\beta(A_0\beta_m + \beta_0 A_m) - 3A_0 A_m = 0 \quad (30)$$

$$A_0 A_m = 0 \quad (31)$$

Further, from the equations (26) and (28), we get

$$\beta_{0m} = 2\beta_{x^m}, \quad (32)$$

$$\beta_0\beta_m = 0. \quad (33)$$

Again, from the equations (32), (33), (27) and (29), we get

$$A_{0m} = 2A_{x^m}, \quad (34)$$

$$A_0\beta_m + \beta_0 A_m = 0. \quad (35)$$

Above discussion leads to the following theorem.

Theorem 3.2 *Let $(M,\bar{F})$ be an n −dimensional Finsler space, where $\bar{F}$ is either of the following:*

1. Exponential-Randers changed metric, i.e., $\bar{F} = Fe^{\beta/F} + \beta$,

2. Infinite series-Randers changed metric, i.e., $\bar{F} = \frac{\beta^2}{\beta - F} + \beta$.

Then $\bar{F}$ is locally dually flat if and only if the following equations are satisfied:

$A_0 A_m = 0, A_{0m} = 2A_{x^m}, \beta_0\beta_m = 0, \beta_{0m} = 2\beta_{x^m}, A_m\beta_0 + A_0\beta_m = 0.$

4. Conclusions

The all above discussion concludes that

1. If Matsumoto-Randers changed Finsler square root metric locally dually flat then so are exponential-Randers changed Finsler square root metric and infinite series-Randers changed Finsler square root metric.
2. If exponential-Randers changed Finsler square root metric is locally dually flat then so is infinite series-Randers changed Finsler square root metric and vice-versa.

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