SOME PROPERTIES OF $m$–PROJECTIVE CURVATURE TENSOR IN A P-SASAKIAN MANIFOLD

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Abstract: The object of present paper is to study some properties of $m$–projective curvature tensor in a $P$-Sasakian manifold.

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1. Introduction

Sato[12,13] introduced the notation of an (almost) para-contact structure, either $P$-Sasakian manifold or $SP$-Sasakian manifold and obtain very interesting results about such manifolds. In 1970, Pokhariyal and Mishra [10] defined tensor fields $W^*$ in the Riemannian manifold known as $m$-projective curvature tensor. Later, Ojha [7,8] defined and studied some properties of this curvature tensor in a Sasakian manifolds and Kähler manifolds. Singh[14], Taleshian and N. Asghari [12] studied $m$-projective curvature tensor in $P$-Sasakian manifold. Also $m$-projective curvature tensor have been studied by Chaubey and Ojha [4], Chaubey [3], Prakash et al. [11], Singh et al. 15,16] and many others in different structures.

The paper is organized as follows. In section 2, we give a brief account of $P$- Sasakian manifold. In section 3 we study $m$-projectively semi-symmetric $P$-Sasakian manifold. Section 4 deals with $\xi$- $m$-projectively flat $P$-Sasakian manifold. In section 5 we study quasi $m$-projectively flat $P$-Sasakian manifold. In the next section we proved that $m$-projectively recurrent $P$- Sasakian manifold is $m$-projectively semi-symmetric $P$- Sasakian manifold and hence $P$- Sasakian manifold is $SP$-Sasakian manifold. The last section deals with an Einstein $P$-Sasakian manifold satisfying$(divW^*)(X,Y)Z = 0$.
2. P-Sasakian Manifold

Let $M$ be an $n$-dimensional differentiable manifold on which there exists a $(1,1)$ tensor field $\phi$, a vector field $\xi$ and $1$ from $\eta$ satisfying

\begin{align*}
\phi^2 &= X - \eta(X)\xi, \\
\eta(\xi) &= 1, \\
\eta\phi &= 0, \\
\phi\xi &= 0
\end{align*}

is called an almost para contact manifold and the structure $(\phi, \xi, \eta)$ is called an almost para contact structure.

The first and one of the remaining last three above relations imply the other two relations. Let $g$ be a compatible Riemannian metric with $(\phi, \xi, \eta)$-structure such that

\begin{align*}
g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y) \tag{5} \\
\text{or, equivalently,} \\
g(\phi X, Y) &= g(X, \phi Y) \quad \text{and} \quad g(X, \xi) = \eta(X) 
\end{align*}

for all $X, Y \in TM$.

Then $M$ is called an almost para contact Riemannian manifold or an almost para contact metric manifold with an almost para contact Riemannian structure $(\phi, \xi, \eta, g)$.

**Definition:** An almost para contact Riemannian manifold is called P-Sasakian manifold if

\begin{align*}
(\nabla_X \phi)(Y) &= -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi 
\end{align*}

for all $X, Y \in TM$.

where $\nabla$ denotes the operator of co-variant differentiation with respect to Riemannian metric $g$.

On P-Sasakian manifold, we have

\begin{align*}
(\nabla_X \eta)(Y) &= g(\phi X, Y) = (\nabla_Y \eta)(X) \\
(\nabla_X \eta)(Y) &= \Phi(X, Y) \quad \text{where} \quad \Phi(X, Y) \equiv g(\phi X, Y) \\
(\nabla_X \xi) &= \phi X
\end{align*}

Also in an P-Sasakian manifold $M$, the curvature tensor $R$, the Ricci tensor $S$, and the Ricci operator $Q$ satisfy

\begin{align*}
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X \tag{11} \\
R(\xi, X)Y &= \eta(Y)X - g(X, Y)\xi \tag{12} \\
\eta(R(X, Y)Z) &= g(X, Z)\eta(Y) - g(Y, Z)\eta(X) \tag{13}
\end{align*}
Some Properties of m-projective ...

(S, Y, X) = −(n − 1)η(X)
Qξ = −(n − 1)ξ
S(ϕX, ϕY) = S(X, Y) + (n − 1)η(X)η(Y)
S(X, ϕY) = S(X, ϕY)

**Definition:** An almost paracontact Riemannian manifold is said to be η-Einstein [17] if the Ricci tensor S satisfy

\[ S(X, Y) = a g(X, Y) + b η(X)η(Y) \]  \hspace{1cm} (18)

Where \( a \) and \( b \) are smooth functions on the manifold. In particular, if \( b = 0 \), then \( M \) is an Einstein manifold.

Pokhariyal[9], Pokhariyal and Mishra [10] have defined a tensor field \( W^* \) on a Riemannian manifold as

\[ W^*(X, Y, Z) = R(X, Y)Z + \frac{1}{2(n−1)} \left[ S(Y, Z)X − S(X, Z)Y + g(Y, Z)QX − g(X, Z)QY \right] \]  \hspace{1cm} (19)

for arbitrary vector fields \( X, Y, Z \) where \( S \) is Ricci tensor of type (0,2) so that

\[ W^*(X, Y, Z, U) = g(W^*(X, Y)Z, U) = W^*(Z, U, X, Y) \]

Such a tensor field \( W^* \) is called m-projective curvature tensor.

It can be easily shown that in a \( P \)-Sasakian manifold the \( m − \)projective curvature tensor \( W^* \) satisfies the following relations

\[ W^*(X, Y)ξ = \frac{1}{2} [η(X)Y − η(Y)X] − \frac{1}{2(n−1)} [η(Y)QX − η(X)QY] \]  \hspace{1cm} (20)

\[ η(W^*(X, Y)ξ) = 0 \]  \hspace{1cm} (21)

\[ W^*(ξ, Y)Z = \frac{1}{2} [η(Z)Y − g(Y, Z)ξ] − \frac{1}{2(n−1)} [S(Y, Z)ξ − η(Z)QY] \]  \hspace{1cm} (22)

\[ η(W^*(ξ, Y)Z) = −\frac{1}{2} \left[ g(Y, Z) + \frac{1}{(n−1)} S(Y, Z) \right] \]  \hspace{1cm} (23)

\[ η(W^*(X, Y)Z) = \frac{1}{2} [g(X, Z)η(Y) − g(Y, Z)η(X)] − \frac{1}{2(n−1)} [S(Y, Z)η(X) − S(X, Z)η(Y)] \]  \hspace{1cm} (24)

3. \( m \)-Projectively Semi-symmetric \( P \)-Sasakian Manifold

**Definition:** An \( n \)-dimensional \( P \)-Sasakian manifold is called \( W^* \)-semi-symmetric if it satisfies

\[ R(X, Y).W^* = 0 \]  \hspace{1cm} (25)

Where \( R \) is the Riemannian curvature tensor and to be considered as a derivation of the tensor algebra at each point of the manifold for tangents vectors \( X, Y \).
From (25), we have
\( (R(\xi, X).W^*)(Y, Z) = 0 \)

(26)

The above equation can be written as

(27)

Using (12) in (27), we have
\[ \eta(W^*(Y, Z)U)X - \eta(Y)W^*(X, Z)U + g(X, Y)W^*(\xi, Z)U - \eta(Z)W^*(Y, X)U + g(X, Z)W^*(\xi, U) - \eta(U)W^*(Y, Z)X + g(X, U)W^*(Y, Z)\xi = 0 \]

(28)

Taking inner product of above equation with \( \xi \) and using (2), we have
\[ \eta(W^*(Y, Z)U)\eta(X) - \eta(Y)\eta(W^*(X, Z)U) + g(X, Y)\eta(W^*(\xi, Z)U) - \eta(Z)\eta(W^*(Y, X)U) + g(X, Z)\eta(W^*(Y, \xi)U) - \eta(U)\eta(W^*(Y, Z)X) + g(X, U)\eta(W^*(Y, Z)\xi) = 0 \]

(29)

using (19), (21), (23) and (24), we have
\[-R(Y, Z, U, X) - \frac{1}{2} [g(X, Y)g(Z, U) - g(X, Z)g(Y, U) + g(X, Y)\eta(Z)\eta(U) - g(X, Z)\eta(Y)\eta(U)] + \frac{1}{2(n-1)} [g(Z, U)S(Y, X) - g(Y, U)S(Z, X) + S(X, Z)\eta(Y)\eta(U) - S(X, Y)\eta(Z)\eta(U)] = 0 \]

(30)

Putting \( Z = U = e_i \) in the above equation and taking summation over \( i, 1 \leq i \leq n \),

We get
\[ S(X, Y) = (1 - n)g(X, Y) \]

(31)

Thus we can state the theorem:

**Theorem 3.1:** An \( m \) -projectively semi-symmetric \( P \)-Sasakian manifold is an Einstein manifold

Using (31) in (30), we have
\[ R(Y, Z, U, X) = g(Y, U)g(X, Z) - g(Z, U)g(X, Y) \]

This gives
\[ R(Y, Z)U = -[g(Z, U)Y - g(Y, U)Z] \]

(32)

The above equation show that \( M \) is of constant curvature \(-1\) and consequently it is locally isometric with the hyperbolic space \( H^n(-1) \). Thus we have the following theorem:

**Theorem 3.2:** An \( n \)-dimensional \( P \)-Sasakian manifold \( M \) is \( m \) -projectively semi-symmetric, then it is locally isometric to the hyperbolic space \( H^n(-1) \).
Some Properties of $m$-projective $P$-Sasakian manifold is of constant curvature. But it is known [1,2] that if a $P$-Sasakian manifold is of constant curvature, then it is an $SP$-Sasakian manifold. Thus we can state theorem

**Theorem 3.3:** An $m$-projectively semi-symmetric $P$-Sasakian manifold is an $SP$-Sasakian manifold.

### 4. $\xi$-$m$-Projectively Flat $P$-Sasakian Manifold

**Definition:** An $n$-dimensional $P$-Sasakian manifold $M$ is said to be $\xi$-$m$–projectively flat [18] if $W^*(X,Y)\xi = 0$ for all $X, Y \in TM$.

Let $W^*(X,Y)\xi = 0$, then from (20), we obtain

$$\frac{1}{2} [\eta(X)Y - \eta(Y)X] - \frac{1}{2(n-1)} [\eta(Y)QX - \eta(X)QY] = 0$$  \hspace{1cm} (33)

Putting $Y = \xi$ in (33), we have

$$QX = -(n-1)X$$  \hspace{1cm} (34)

Now taking inner product of above equation with $U$, we get

$$S(X, U) = (1 - n)g(X, U)$$  \hspace{1cm} (35)

Thus $M$ is Einstein manifold. Conversely suppose that(35) is satisfied. Then, by virtue of (34) and (20), we have $W^*(X,Y)\xi = 0$. Thus we have a theorem

**Theorem 4.1:** An $n$-dimensional $P$-Sasakian manifold $M$ is $\xi$-$m$–projectively flat if and only if it is an Einstein manifold.

### 5. Quasi $m$ –Projectively flat $P$-Sasakian Manifold

**Definition:** An $n$-dimensional $P$-Sasakian manifold $M$ is said to be quasi $m$ –projectively flat, if

$$g(W^*(X,Y)Z, \phi U) = 0$$  \hspace{1cm} (36)

for any vector fields $X, Y, Z, U$.

From (19), we have

$$g(W^*(X,Y)Z, \phi U) = g(R(X,Y)Z, \phi U) - \frac{1}{2(n-1)} [S(Y,Z)g(X, \phi U) - S(X,Z)g(Y, \phi U) + g(Y, Z)S(X, \phi U) - g(X, Z)S(Y, \phi U)]$$  \hspace{1cm} (37)

Let $\{e_1, e_2, e_3, \ldots, e_n, \xi\}$ be a local orthonormal basis of vector fields in $M$, by using the fact that $\{\phi e_1, \phi e_2, \phi e_3, \ldots, \phi e_n, \xi\}$ is also a local orthonormal basis, if we put $X = \phi e_i, U = e_i$ in (5.2) and sum up with respect to $i$, then we have
\[
\Sigma_{i=1}^{n-1} g(W^*(\phi e_i, Y)Z, \phi e_i) = \Sigma_{i=1}^{n-1} g(R(\phi e_i, Y)Z, \phi e_i) - \frac{1}{2(n-1)} \Sigma_{i=1}^{n-1} [S(Y, Z)g(\phi e_i, \phi e_i) - S(\phi e_i, Z)g(Y, \phi e_i) + g(Y, Z)S(\phi e_i, \phi e_i) - g(\phi e_i, Z)S(Y, \phi e_i)]
\]

(38)

On a \(P\)-Sasakian manifold, given by De and Sarkar [5], Matsumoto [6], we have

\[
\Sigma_{i=1}^{n-1} g(\phi e_i, Y)g(\phi e_i, \phi e_i) = S(\phi Y, \phi Z) + g(\phi Y, \phi Z)
\]

(39)

\[
\Sigma_{i=1}^{n-1} S(\phi e_i, \phi e_i) = r + (n - 1)
\]

(40)

\[
\Sigma_{i=1}^{n-1} g(\phi e_i, \phi e_i) = S(\phi Y, \phi Z)
\]

(41)

\[
\Sigma_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n - 1)
\]

(42)

\[
\Sigma_{i=1}^{n-1} S(\phi e_i, Z)g(Y, \phi e_i) = S(Y, Z) - S(Z, \xi)e(Y) = S(Y, Z) + (n - 1)e(Y)e(Z)
\]

(43)

\[
\Sigma_{i=1}^{n-1} R(e_i, Y, Z, e_i) = \Sigma_{i=1}^{n-1} R(\phi e_i, Y, Z, \phi e_i) = S(Y, Z) + g(\phi Y, \phi Z)
\]

(44)

Using (39) – (44) in (38), we get

\[
\Sigma_{i=1}^{n-1} g(W^*(\phi e_i, Y)Z, \phi e_i) = \frac{1}{2(n-1)} [(n + 1)S(Y, Z) + (n - 1 - r)g(Y, Z)]
\]

(45)

If \(M\) is quasi \(m\) –projectively flat, then (35) reduces to

\[(n + 1)S(Y, Z) = [r + (n - 1)]g(Y, Z)
\]

(46)

Putting \(Z = \xi\) in (46) and then using (6) and (11), we have

\[r = -n(m - 1)
\]

(47)

Using (47) in (46), we get

\[S(Y, Z) = (1 - n)g(Y, Z)
\]

(48)

Therefore \(M\) is Einstein manifold. Thus we have a theorem

**Theorem 5.1:** A quasi \(m\) –projectively flat \(P\)-Sasakian manifold is Einstein manifold.

Now using (48) in (19), we get

\[W^*(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y
\]

(49)

If \(P\)-Sasakian manifold is \(m\) –projectively flat, i.e., \(W^*(X, Y)Z = 0\) then from (49), we have

\[R(X, Y)Z = -[g(Y, Z)X - g(X, Z)Y]
\]

Hence we can state the following theorem

**Theorem 5.2:** A quasi \(m\) –projectively flat \(P\)-Sasakian manifold is locally isometric to the hyperbolic space \(H^n(-1)\) if and only if \(M\) is \(m\) –projectively flat.

6. \(m\) –Projectively Recurrent \(P\)-Sasakian Manifold

**Definition:** An \(n\)-dimensional \(P\)-Sasakian manifold \(M\) is said to be \(m\) –projectively recurrent if it satisfies
Some Properties of m-projective ...

\[(\nabla_U W^*)(X, Y)Z = A(U)W^*(X, Y)Z\]  
for some non-zero 1-form \(A\).

We define a function

\[f^2 = g(W^*, W^*)\]  
Now using the fact that \(\nabla_U g = 0\), (51) gives that \(f(Uf) = f^2(A(U))\)

Since \(f \neq 0\), we have

\[Uf = f(A(U))\]  
From the above equation, we have

\[X(Uf) - U(Xf) = \{XA(U) - UA(X)\}f\]  
Therefore

\[(\nabla_X \nabla_U - \nabla_U \nabla_X - \nabla_{[X,U]} f) = \{XA(U) - UA(X) - A([X,U])\}f = [dA(X, U)]f\]  
Since the left hand side of (54) is zero and \(f \neq 0\), we obtain

\[dA(X, U) = 0\]  
that is 1-form \(A\) is closed.

From (50), we have

\[(\nabla_V \nabla_U W^*)(X, Y)Z = [VA(U) + A(V)A(U)]W^*(X, Y)Z\]  
In view of (55) and (56), we have

\[(\nabla_V \nabla_U W^*)(X, Y)Z - (\nabla_U \nabla_V W^*)(X, Y)Z - (\nabla_{[V,U]} W^*)(X, Y)Z = 2dA(V, U)W^*(X, Y)Z = 0\]

Hence \((R(V, U).W^*)(X, Y)Z = 0\), where \(R(V, U)\) is to be considered as a derivation of tensor algebra at each point of the manifold for tangent vectors \(V, U\). Thus the \(m\) --projectively recurrent is \(m\) --projectively semi-symmetric. Thus from theorem [3.3], we state the following theorem

**Theorem 6.1:** An \(m\) --projectively recurrent \(P\)-Sasakian manifold is an \(SP\)-Sasakian manifold.

**7. Einstein \(P\)-Sasakian manifold satisfying \(div W^*(X, Y)Z = 0\)**

From (19), we have

\[W^*(X, Y, Z, V) = \{R(X, Y, Z, V) - \frac{1}{2(r-1)}[S(Y, Z)g(X, V) - S(X, Z)g(Y, V) + g(Y, Z)S(X, V) - g(X, Z)S(Y, V)]\]  
Now differentiating above equation covariantly, we have
\[
(V_0 W^*)(X,Y,Z,V) = (V_0 R)(X,Y,Z,V) - \frac{1}{2(n-1)}[(V_0 S)(Y,Z)g(X,V) - (V_0 S)(X,Z)g(Y,V) + g(Y,Z)(V_0 S)(X,V) - g(X,Z)(V_0 S)(Y,V)]
\] (58)

Putting \( U = V = e_i \) in (58) and summing over \( i, 1 \leq i \leq n \), we have

\[
(div W^*)(X,Y)Z = (div R)(X,Y)Z - \frac{1}{2(n-1)}[(V_0 S)(Y,Z) - (V_0 S)(X,Z) + \frac{1}{2} g(Y,Z)dr(X) - \frac{1}{2} g(X,Z)dr(Y)]
\] (59)

On a \( P \)-Sasakian manifold

\[
(div R)(X,Y)Z = (V_0 S)(Y,Z) - (V_0 S)(X,Z)
\] (60)

Using (60) in (59), we have

\[
(div W^*)(X,Y)Z = \frac{2(n-3)}{2(n-1)}[(V_0 S)(Y,Z) - (V_0 S)(X,Z)] - \frac{1}{4(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]
\] (61)

For Einstein \( P \)-Sasakian manifold, we have

\[
(V_0 S)(Y,Z) = 0
\] (62)

Using (62) in (61), we have

\[
(div W^*)(X,Y)Z = -\frac{1}{4(n-1)}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]
\] (63)

If \( div W^* = 0 \), then from (63), we get

\[
g(Y,Z)dr(X) - g(X,Z)dr(Y) = 0
\]

This shows that \( g(X,Z)dr(Y) = 0 \), therefore \( dr = 0 \), that is \( r \) is constant.

Conversely, if \( r \) is constant then from (63), we get

\[
(div W^*)(X,Y)Z = 0
\]

Thus we can state theorem

**Theorem 7.1:** An Einstein \( P \)-Sasakian manifold satisfying \( (div W^*)(X,Y)Z = 0 \) if and only if the scalar curvature \( r \) is covariant constant.

Since a manifold is said to be \( m \) –projectively conservative if \( div W^* = 0 \)

Thus we have the following theorem as a corollary of the theorem [7.1]

**Theorem 7.2:** An Einstein \( P \)-Sasakian manifold \( M \) of dimension \( n(n > 3) \) is \( m \) – projectively conservative if and only if the scalar curvature is constant.

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