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SUM OF THREE AND MORE TRIANGULAR RANDOM VARIABLES

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Abstract : In this paper, we have derived the probability density function (pdf) for the sum of three independent triangular random variables with the findings of several cases and sub cases. The pdf for one such sub case is deduced and pdf in other cases in tabular form are shown. The pdf regarding n-product of triangular random variables is also evaluated.

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1. Introduction

The distribution of sum of random variables has a wide variety of applications in several fields such as application of the sum of independent Gamma random variables in the problems of queuing theory, determining of total waiting time of total excess water flow in a dam, in civil engineering etc.

In 1999, Loaiciga and Leipnik [12] derived the probability distribution of sum of two Gumbel random variables and gave several examples of its application in hydrology. The contribution of other researchers regarding to the distribution of sum of random variables can be found in Agrawal and Elmaghraby [1], Albert [2], Aroian [3], Cramer [5,6], Holm and Alouini [10], Kotz and Van Dorp [11] Lukacs [13], Lukacs and Laha [14], Moschopoulos [15], Nason [16] as well Witner [18] which are worth mentioning. We shall extend the work of Garg, Choudhary and Kalla [9] by deriving the pdf of the sum of three non identical, independent and triangular random variables lying on different supports. For the distribution of sum of triangular and bilateral exponential random variables (see, e.g.[8]).

The triangular distribution is popular for using in modeling estimation of some uncertain quantity in business risk models, oil and gas exploration, business decision making based on simulation of the outcome, in project management to model events during an interval and in audio dithering. The advantages of using triangular distribution over Beta distribution have been discussed by Van Dorp and Kotz [17]. Recent popularity of the triangular distribution can be attributed to its use in discrete system simulation [4], Monte

Carlo simulation technique [19] and its use in standard uncertainty analysis software - such as @Risk (developed by the Palisade Corporation) or Crystal Ball (developed by Decision Engineering).

2. Distribution of Pdf of Sum of Three Triangular Random Variables

Taking $a_1 > 0, b_1 > 0$ as lower and upper limits respectively with mode $m_1 > 0$ and a random variable X is a continuous probability distribution of a triangular distribution is defined as

$$f_1(x) = \begin{cases} \dfrac{2(x-a_1)}{(b_1-a_1)(m_1-a_1)}, & a_1 \le x \le m_1 \\ \\ \dfrac{2(b_1-x)}{(b_1-a_1)(b_1-m_1)}, & m_1 \le x \le b_1 \end{cases} \tag{1}$$

We consider two more independent and triangularly distributed random variables Y and Z on the supports $a_2 \le y \le b_2$ and $a_3 \le z \le b_3$ having modes m_2 *and* m_3 respectively.

The application of Laplace transform with its inversion the pdf $h(w)$ of the sum of three random variables $W = X + Y + Z$ is derived as (Springer [20])

$$h(w) = L_s^{-1}\left[L_s f_1(x_1) L_s f_2(x_2) L_s f_3(x_3)\right], \tag{2}$$

where $f_1(x_1)$, $f_2(x_2)$, and $f_3(x_3)$ stands for the pdf of random variables X, Y and Z respectively.

The Laplace transform of $f_1(x)$ can easily derived

$$L_s[f_1(x)] = \begin{cases} \dfrac{2}{(b_1-a_1)(m_1-a_1)}\left((m_1-a_1)\dfrac{e^{-sm_1}}{-s} - \dfrac{e^{-sm_1}}{s^2} + \dfrac{e^{-sa_1}}{s^2}\right), a_1 \le x \le m_1 \\ \dfrac{2}{(b_1-a_1)(b_1-m_1)}\left((b_1-m_1)\dfrac{e^{-sm_1}}{s} - \dfrac{e^{-sm_1}}{s^2} + \dfrac{e^{-sb_1}}{s^2}\right), m_1 \le x \le b_1 . \end{cases} \tag{3}$$

In the similar manner, we can have Laplace transform of $f_2(x)$ and $f_3(x)$ are as follows

$$L_s[f_2(y)] = \begin{cases} \dfrac{2}{(b_2-a_2)(m_2-a_2)}\left((m_2-a_2)\dfrac{e^{-sm_2}}{-s} - \dfrac{e^{-sm_2}}{s^2} + \dfrac{e^{-sa_2}}{s^2}\right), a_2 \le y \le m_2 \\ \dfrac{2}{(b_2-a_2)(b_2-m_2)}\left((b_2-m_2)\dfrac{e^{-sm_2}}{s} - \dfrac{e^{-sm_2}}{s^2} + \dfrac{e^{-sb_2}}{s^2}\right), m_2 \le y \le b_2. \end{cases} \tag{4}$$

and

$$L_s[f_3(y)] = \begin{cases} \dfrac{2}{(b_3-a_3)(m_3-a_3)}\left((m_3-a_3)\dfrac{e^{-sm_3}}{-s} - \dfrac{e^{-sm_3}}{s^2} + \dfrac{e^{-sa_3}}{s^2}\right), a_3 \le y \le m_3 \\ \dfrac{2}{(b_3-a_3)(b_3-m_3)}\left((b_3-m_3)\dfrac{e^{-sm_3}}{s} - \dfrac{e^{-sm_3}}{s^2} + \dfrac{e^{-sb_3}}{s^2}\right), m_3 \le y \le b_3. \end{cases} \quad (5)$$

Now, evaluation of pdf $h(w)$ needs consideration of the following eight different cases where the values of x, y, and z are given in different segments of their respective domains

I. $a_1 \le x \le m_1,\ a_2 \le y \le m_2,\ a_3 \le z \le m_3$,
II. $a_1 \le x \le m_1,\ a_2 \le y \le m_2,\ m_3 \le z \le b_3$,
III. $a_1 \le x \le m_1,\ m_2 \le y \le b_2,\ a_3 \le z \le m_3$,
IV. $a_1 \le x \le m_1,\ m_2 \le y \le b_2,\ m_3 \le z \le b_3$,
V. $m_1 \le x \le b_1,\ a_2 \le y \le m_2,\ a_3 \le z \le m_3$,
VI. $m_1 \le x \le b_1,\ a_2 \le y \le m_2,\ m_3 \le z \le b_3$,
VII. $m_1 \le x \le b_1,\ m_2 \le y \le b_2,\ a_3 \le z \le m_3$,
VIII. $m_1 \le x \le b_1,\ m_2 \le y \le b_2,\ m_3 \le z \le b_3$.

CASE II

For $a_1 \le x_1 \le m_1$, $a_2 \le y \le m_2$, and $m_3 \le z \le b_3$ and let $h_2(w)$ be the value of $h(w)$ in Case II, applying appropriate part of equations (3) to (5) in equation (2), we obtain

$$h_2(w) = L_s^{-1}\left[K_2\left\{\left((m_1-a_1)\frac{e^{-sm_1}}{-s} - \frac{e^{-sm_1}}{-s^2} + \frac{e^{-sa_1}}{s^2}\right)\right\} \cdot\left\{\left((m_2-a_2)\frac{e^{-sm_2}}{-s} - \frac{e^{-sm_2}}{-s^2} + \frac{e^{-sa_2}}{s^2}\right)\right\} \cdot\left\{\left((b_3-m_3)\frac{e^{-sm_3}}{-s} - \frac{e^{-sm_3}}{-s^2} + \frac{e^{-sb_3}}{s^2}\right)\right\}\right], \quad (6)$$

where $K_2 = \dfrac{8}{(b_1-a_1)(b_2-a_2)(b_3-a_3)(m_1-a_1)(m_2-a_2)(b_3-m_3)}$.

Clearly, the right-hand side of equation (6) contains 27 terms and the inverse Laplace transform of these are evaluated by partial fractions, on using the following result [31, p.137 (5), p.133 (3)]

$$L_s^{-1}\left[e^{-as}\varphi(s);x\right]=f(x-a)H(x-a), \tag{7}$$

where is the well known Heaviside unit step function.

$$L_s^{-1}\left(\frac{1}{s^{n+1}}\;;x\right)=\frac{x^n}{n!}\ , \tag{8}$$

and thus the results obtained are as follows

$$L_S^{-1}\left\{(m_1-a_1)(m_2-a_2)(b_3-m_3)\frac{e^{-s(m_1+m_2+m_3)}}{s^3};w\right\}$$
$$=(m_1-a_1)(m_2-a_2)(b_3-m_3)\frac{[w-(m_1+m_2+m_3)]^2}{2!}\ ,\ w>m_1+m_2+m_3 \tag{9}$$

$$L_S^{-1}\left\{(m_1-a_1)(b_3-m_3)\frac{e^{-s(m_1+m_2+m_3)}}{s^4}\;;w\right\}$$
$$=(m_1-a_1)(b_3-m_3)\frac{[w-(m_1+m_2+m_3)]^3}{3!}\ ,\qquad w>m_1+m_2+m_3 \tag{10}$$

$$L_S^{-1}\left\{(m_1-a_1)(b_3-m_3)\frac{e^{-s(m_1+a_2+m_3)}}{-s^4};w\right\}$$
$$=(m_1-a_1)(b_3-m_3)\frac{[w-(m_1+a_2+m_3)]^3}{3!}\ ,\quad w>m_1+a_2+m_3 \tag{11}$$

$$L_S^{-1}\left\{(m_2-a_2)(b_3-m_3)\frac{e^{-s(m_1+m_2+m_3)}}{s^4};w\right\}$$
$$=(m_2-a_2)(b_3-m_3)\frac{[w-(m_1+m_2+m_3)]^3}{3!}\ ,\quad w>m_1+m_2+m_3 \tag{12}$$

$$L_S^{-1}\left\{(b_3-m_3)\frac{e^{-s(m_1+m_2+m_3)}}{s^5};w\right\}=(b_3-m_3)\frac{[w-(m_1+m_2+m_3)]^4}{4!}\ , \tag{13}$$
$$w>m_1+m_2+m_3$$

$$L_S^{-1}\left\{-(b_3-m_3)\frac{e^{-s(m_1+a_2+m_3)}}{s^5}\;;w\right\}=-(b_3-m_3)\frac{[w-(m_1+a_2+m_3)]^4}{4!}\ , \tag{14}$$
$$w>m_1+a_2+m_3$$

$$L_S^{-1}\left\{(m_2-a_2)(b_3-m_3)\frac{e^{-s(a_1+m_2+m_3)}}{-s^4};w\right\}$$
$$=-(m_2-a_2)(b_3-m_3)\frac{[w-(a_1+m_2+m_3)]^3}{3!}\ ,\quad w>a_1+m_2+m_3 \tag{15}$$

$$L_S^{-1}\left\{-(b_3-m_3)\frac{e^{-s(a_1+m_2+m_3)}}{-s^5};w\right\}=-(b_3-m_3)\frac{[w-(a_1+m_2+m_3)]^4}{4!}\ , \tag{16}$$
$$w>a_1+m_2+m_3$$

$$L_S^{-1}\left\{(b_3-m_3)\frac{e^{-s(a_1+a_2+m_3)}}{s^5};w\right\}=(b_3-m_3)\frac{[w-(a_1+a_2+m_3)]^4}{4!}\ , \tag{17}$$
$$w>a_1+a_2+m_3$$

$$L_S^{-1}\left\{(m_1-a_1)(m_2-a_2)\frac{e^{-s(m_1+m_2+m_3)}}{-s^4};w\right\}$$
$$=-(m_1-a_1)(m_2-a_2)\frac{[w-(m_1+m_2+m_3)]^3}{3!}\ ,\ w>m_1+m_2+m_3 \tag{18}$$

$$L_S^{-1}\left\{-(m_1-a_1)\frac{e^{-s(m_1+m_2+m_3)}}{s^5};w\right\}=-(m_1-a_1)\frac{[w-(m_1+m_2+m_3)]^4}{4!}\ , \tag{19}$$
$$w>m_1+m_2+m_3$$

$$L_S^{-1}\left\{(m_1-a_1)\frac{e^{-s(m_1+a_2+m_3)}}{s^5};w\right\}=(m_1-a_1)\frac{[w-(m_1+a_2+m_3)]^4}{4!}\ , \tag{20}$$
$$w>m_1+a_2+m_3$$

$$L_S^{-1}\left\{-(m_2-a_2)\frac{e^{-s(m_1+m_2+m_3)}}{s^5};w\right\}=-(m_2-a_2)\frac{[w-(m_1+m_2+m_3)]^4}{4!}\ , \tag{21}$$
$$w>m_1+m_2+m_3$$

$$L_S^{-1}\left\{\frac{e^{-s(m_1+m_2+m_3)}}{-s^6};w\right\}=-\frac{[w-(m_1+m_2+m_3)]^5}{5!}\ ,w>m_1+m_2+m_3 \tag{22}$$

$$L_S^{-1}\left\{\frac{e^{-s(m_1+a_2+m_3)}}{s^6};w\right\}=\frac{[w-(m_1+a_2+m_3)]^5}{5!}\ ,w>m_1+a_2+m_3 \tag{23}$$

$$L_S^{-1}\left\{(m_2-a_2)\frac{e^{-s(a_1+m_2+m_3)}}{s^5};w\right\}=(m_2-a_2)\frac{[w-(a_1+m_2+m_3)]^4}{4!}, \quad w>a_1+m_2+m_3 \tag{24}$$

$$L_S^{-1}\left\{\frac{e^{-s(a_1+m_2+m_3)}}{s^6};w\right\}=\frac{[w-(a_1+m_2+m_3)]^5}{5!}, w>a_1+m_2+m_3 \tag{25}$$

$$L_S^{-1}\left\{\frac{e^{-s(a_1+a_2+m_3)}}{-s^6};w\right\}=-\frac{[w-(a_1+a_2+m_3)]^5}{5!}, w>a_1+a_2+m_3 \tag{26}$$

$$L_S^{-1}\left\{(m_1-a_1)(m_2-a_2)\frac{e^{-s(m_1+m_2+b_3)}}{s^4};w\right\} \tag{27}$$

$$=(m_1-a_1)(m_2-a_2)\frac{[w-(m_1+m_2+b_3)]^3}{3!}, \quad w>m_1+m_2+b_3$$

$$L_S^{-1}\left\{(m_1-a_1)\frac{e^{-s(m_1+m_2+b_3)}}{s^5};w\right\}=(m_1-a_1)\frac{[w-(m_1+m_2+b_3)]^4}{4!}, w>m_1+m_2+b_3 \tag{28}$$

$$L_S^{-1}\left\{-(m_1-a_1)\frac{e^{-s(m_1+a_2+b_3)}}{s^5};w\right\}=-(m_1-a_1)\frac{[w-(m_1+a_2+b_3)]^4}{4!}, w>m_1+a_2+b_3 \tag{29}$$

$$L_S^{-1}\left\{(m_2-a_2)\frac{e^{-s(m_1+m_2+b_3)}}{s^5};w\right\}=(m_2-a_2)\frac{[w-(m_1+m_2+b_3)]^4}{4!}, w>m_1+m_2+b_3 \tag{30}$$

$$L_S^{-1}\left\{\frac{e^{-s(m_1+m_2+b_3)}}{s^6};w\right\}=\frac{[w-(m_1+m_2+b_3)]^5}{5!}, \quad w>m_1+m_2+b_3 \tag{31}$$

$$L_S^{-1}\left\{-\frac{e^{-s(m_1+a_2+b_3)}}{s^6};w\right\}=-\frac{[w-(m_1+a_2+b_3)]^5}{5!}, \quad w>m_1+a_2+b_3 \tag{32}$$

$$L_S^{-1}\left\{-(m_2-a_2)\frac{e^{-s(a_1+m_2+b_3)}}{s^5};w\right\}=-(m_2-a_2)\frac{[w-(a_1+m_2+b_3)]^4}{4!}, w>m_1+a_2+b_3 \tag{33}$$

$$L_S^{-1}\left\{-\frac{e^{-s(a_1+m_2+b_3)}}{s^6};w\right\}=-\frac{[w-(a_1+m_2+b_3)]^5}{5!}, \quad w>a_1+m_2+b_3 \tag{34}$$

$$L_S^{-1}\left\{\frac{e^{-s(a_1+a_2+b_3)}}{s^6};w\right\}=-\frac{[w-(a_1+a_2+b_3)]^5}{5!}, \quad w>a_1+a_2+b_3 \tag{35}$$

3. Calculation of pdf $h_2(w)$ for Different Ranges of Values of w

The value of $h_2(w)$ from equation (6) is expressed for different values of w by using the results given in equations from (9) to (35). The value of pdf $h_2(w)$ is derived on combining equations (9), (10), (12), (13), (18), (19), (21) and (22) for $w > m_1 + m_2 + m_3$, we find

$$h_2(w) == -k_2 \left\{ [(m_1 - a_1)(m_2 - a_2)(m_3 - b_3)] \frac{[w - (m_1 + m_2 + m_3)]^2}{2!} + [(m_1 - a_1)(m_2 - a_2) + (m_2 - a_2)(m_3 - b_3) + (m_3 - b_3)(m_1 - a_1)] \frac{[w - (m_1 + m_2 + m_3)]^3}{3!} + [(m_1 - a_1) + (m_2 - a_2) + (m_3 - b_3)] \frac{[w - (m_1 + m_2 + m_3)]^4}{4!} + \frac{[w - (m_1 + m_2 + m_3)]^5}{5!} \right\}, \tag{36}$$

for $w > m_1 + m_2 + m_3$

Proceeding on similar lines we can obtain $h_2(w)$ for other different values of w as follows

$$h_2(w) = k_2 \left\{ [(m_2 - a_2)(m_3 - b_3)] \frac{[w - (a_1 + m_2 + m_3)]^3}{3!} + [(m_2 - a_2) + (m_3 - b_3)] \frac{[w - (a_1 + m_2 + m_3)]^4}{4!} + \frac{[w - (a_1 + m_2 + m_3)]^5}{5!} \right\}, \tag{37}$$

for $w > a_1 + m_2 + m_3$

$$h_2(w) = k_2 \left\{ [(m_1 - a_1)(m_3 - b_3)] \frac{[w - (m_1 + a_2 + m_3)]^3}{3!} + [(m_1 - a_1) + (m_3 - b_3)] \frac{[w - (m_1 + a_1 + m_3)]^4}{4!} + \frac{[w - (m_1 + a_2 + m_3)]^5}{5!} \right\}, \tag{38}$$

for $w > m_1 + a_2 + m_3$

$$h_2(w) = k_2 \left\{ [(m_1 - a_1)(m_2 - a_2)] \frac{[w - (m_1 + m_2 + b_3)]^3}{3!} + [(m_1 - a_1) + (m_2 - a_2)] \frac{[w - (m_1 + m_2 + b_3)]^4}{4!} + \frac{[w - (m_1 + m_2 + b_3)]^5}{5!} \right\}, \tag{39}$$

for $w > m_1 + m_2 + b_3$

$$h_2(w) = -k_2 \left\{ (m_3 - b_3) \frac{[w - (a_1 + a_2 + m_3)]^4}{4!} + \frac{[w - (a_1 + a_2 + m_3)]^5}{5!} \right\}, \tag{40}$$

for $w > a_1 + a_2 + m_3$

$$h_2(w) = -k_2\left\{(m_1 - a_1)\frac{\left[w-(m_1+a_2+b_3)\right]^4}{4!} + \frac{\left[w-(m_1+a_2+b_3)\right]^5}{5!}\right\}, \quad (41)$$

for $w > m_1 + a_2 + b_3$

$$h_2(w) = -k_2\left\{(m_2 - a_2)\frac{\left[w-(a_1+m_2+b_3)\right]^4}{4!} + \frac{\left[w-(a_1+m_2+b_3)\right]^5}{5!}\right\}, \quad (42)$$

for $w > a_1 + m_2 + b_3$

$$h_2(w) = k_2\frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}, \quad (43)$$

for $w > a_1 + a_2 + b_3$

Remark 1. The equations (37) to (43) can be derived from equation (36) on replacing m_1 by a_1 , m_2 by a_2 , m_3 by b_3, m_1, m_2 by a_1, a_2, m_2, m_3 by a_2, b_3, m_1, m_3 by a_1, b_1 and m_1, m_2, m_3 by a_1, a_2, b_3 respectively and each equation is multiplied by a number $(-1)^\alpha$, where α is number of replacement.

Remark 2. The total number of equations in case of three variables is 2^3 and it is easily observed that the number of such equations in case of 'n' variables will be 2^n.

It is observed that in the Case II, the discussion of pdf of sum of two triangular random variables X and Y depend upon the relative magnitudes of $a_1 + m_2$ and $m_1 + a_2$. Thus, the pdf of sum of X + Y + Z is first to be discussed accordingly as the conditions $a_1 + m_2 < m_1 + a_2$, $a_1 + m_2 = m_1 + a_2$, $a_1 + m_2 > m_1 + a_2$.

For $a_1 + m_2 < m_1 + a_2$, we get the following situation for w

$$\left.\begin{array}{l} a_1 + a_2 + m_3 < w < a_1 + a_2 + b_3 \\ a_1 + m_2 + m_3 < w < a_1 + m_2 + b_3 \\ m_1 + a_2 + m_3 < w < m_1 + a_2 + b_3 \\ m_1 + m_2 + m_3 < w < m_1 + m_2 + b_3 \end{array}\right\} \quad \ldots \text{(A)}$$

4. Subcases

Depending upon the relative magnitudes of the points $a_1+a_2+b_3, a_1+m_2+m_3$, $a_1+m_2+b_3, m_1+a_2+m_3$, $m_1+a_2+b_3$, $m_1+m_2+m_3$, we have the following nine **sub cases** :

(i) $a_1+a_2+b_3 < a_1+m_2+m_3, a_1+m_2+b_3 < m_1+a_2+m_3, m_1+a_2+b_3 < m_1+m_2+m_3$

(ii) $a_1+a_2+b_3 < a_1+m_2+m_3, a_1+m_2+b_3 = m_1+a_2+m_3, m_1+a_2+b_3 < m_1+m_2+m_3$

(iii) $a_1+a_2+b_3 < a_1+m_2+m_3, a_1+m_2+b_3 > m_1+a_2+m_3, m_1+a_2+b_3 < m_1+m_2+m_3$

(iv) $a_1+a_2+b_3 = a_1+m_2+m_3, a_1+m_2+b_3 < m_1+a_2+m_3, m_1+a_2+b_3 = m_1+m_2+m_3$

(v) $a_1+a_2+b_3 = a_1+m_2+m_3, a_1+m_2+b_3 = m_1+a_2+m_3, m_1+a_2+b_3 = m_1+m_2+m_3$

(vi) $a_1+a_2+b_3 = a_1+m_2+m_3, a_1+m_2+b_3 > m_1+a_2+m_3, m_1+a_2+b_3 = m_1+m_2+m_3$

(vii) $a_1+a_2+b_3 > a_1+m_2+m_3, a_1+m_2+b_3 < m_1+a_2+m_3, m_1+a_2+b_3 > m_1+m_2+m_3$

(viii) $a_1+a_2+b_3 > a_1+m_2+m_3, a_1+m_2+b_3 = m_1+a_2+m_3, m_1+a_2+b_3 > m_1+m_2+m_3$ and

(ix) $a_1+a_2+b_3 > a_1+m_2+m_3, a_1+m_2+b_3 > m_1+a_2+m_3, m_1+a_2+b_3 > m_1+m_2+m_3$, (44)

Considering $a_1+m_2 = m_1+a_2$, then

$$\left.\begin{aligned} a_1+a_2+m_3 < \ w \ < a_1+a_2+b_3 \\ a_1+m_2+m_3 < \ w \ < a_1+m_2+b_3 \\ m_1+m_2+m_3 < \ w \ < m_1+m_2+b_3 \end{aligned}\right\} \quad \text{... (B)}$$

and the corresponding sub cases are as follows

(i) $a_1+a_2+b_3 \ < \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ < \ m_1+m_2+m_3$

(ii) $a_1+a_2+b_3 \ < \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ = \ m_1+m_2+m_3$

(iii) $a_1+a_2+b_3 \ < \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ > \ m_1+m_2+m_3$

(iv) $a_1+a_2+b_3 \ = \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ < \ m_1+m_2+m_3$

(v) $a_1+a_2+b_3 \ = \ a_1=m_2+m_3, \ \ a_1+m_2+b_3 \ = \ m_1+m_2+m_3$

(vi) $a_1+a_2+b_3 \ = \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ > \ m_1+m_2+m_3$

(vii) $a_1+a_2+b_3 \ > \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ < \ m_1+m_2+m_3$

(viii) $a_1+a_2+b_3 \ > \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ = \ m_1+m_2+m_3$
and

(ix) $a_1+a_2+b_3 \ > \ a_1+m_2+m_3, \ \ a_1+m_2+b_3 \ > \ m_1+m_2+m_3$, (45) Letting $a_1 + m_2 > m_1 + a_2$, then

$$\left.\begin{array}{l} a_1 + a_2 + m_3 < w < a_1 + a_2 + b_3 \\ m_1 + a_2 + m_3 < w < m_1 + a_2 + b_3 \\ a_1 + m_2 + m_3 < w < a_1 + m_2 + b_3 \\ m_1 + m_2 + m_3 < w < m_1 + m_2 + b_3 \end{array}\right\} \quad \text{... (C)}$$

and further sub cases will be as follows

(x) $a_1 + a_2 + b_3 < m_1 + a_2 + m_3, m_1 + a_2 + b_3 < a_1 + m_2 + m_3, a_1 + m_2 + b_3 < m_1 + m_2 + m_3$

(xi) $a_1 + a_2 + b_3 < m_1 + a_2 + m_3, m_1 + a_2 + b_3 = a_1 + m_2 + m_3, a_1 + m_2 + b_3 < m_1 + m_2 + m_3$

(xii) $a_1 + a_2 + b_3 < m_1 + a_2 + m_3, m_1 + a_2 + b_3 > a_1 + m_2 + m_3, a_1 + m_2 + b_3 < m_1 + m_2 + m_3$

(xiii) $a_1 + a_2 + b_3 = m_1 + a_2 + m_3, m_1 + a_2 + b_3 < a_1 + m_2 + m_3, a_1 + m_2 + b_3 = m_1 + m_2 + m_3$

(xiv) $a_1 + a_2 + b_3 = m_1 + a_2 + m_3, m_1 + a_2 + b_3 = a_1 + m_2 + m_3, a_1 + m_2 + b_3 = m_1 + m_2 + m_3$

(xv) $a_1 + a_2 + b_3 = m_1 + a_2 + m_3, m_1 + a_2 + b_3 > a_1 + m_2 + m_3, a_1 + m_2 + b_3 = m_1 + m_2 + m_3$

(xvi) $a_1 + a_2 + b_3 > m_1 + a_2 + m_3, m_1 + a_2 + b_3 < a_1 + m_2 + m_3, a_1 + m_2 + b_3 > m_1 + m_2 + m_3$

(xvii) $a_1 + a_2 + b_3 > m_1 + a_2 + m_3, m_1 + a_2 + b_3 = a_1 + m_2 + m_3, a_1 + m_2 + b_3 > m_1 + m_2 + m_3$ and

(xviii) $a_1 + a_2 + b_3 > m_1 + a_2 + m_3, m_1 + a_2 + b_3 > a_1 + m_2 + m_3, a_1 + m_2 + b_3 > m_1 + m_2 + m_3$, (46)

5. Deduction of Pdf for a Particular Sub Case

To calculate the value of $h_2(w)$ as regard to sub-cases from (i) to (xxvii), from the junction of the equations from (47) to (64) giving pdf for different ranges of w (mention in Table 1) as

$$h_2(w) = -k_2 \left\{ (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3)\right]^4}{4!} + \frac{\left[w - (a_1 + a_2 + m_3)\right]^5}{5!} \right\} \quad (47)$$

$$h_2(w) = -k_2 \left\{ (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3)\right]^4}{4!} + \frac{\left[w - (a_1 + a_2 + m_3)\right]^5}{5!} - \frac{\left[w - (a_1 + a_2 + b_3)\right]^5}{5!} \right\} \quad (48)$$

$$h_2(w) = k_2 \left\{ \left[(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!} + \left[(m_2 - a_2) \right. \right.$$

$$\left. + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!} - (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} \tag{49}$$

$$\left. - \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + a_2 + b_3) \right]^5}{5!} \right\}$$

$$h_2(w) = k_2 \left\{ \left[(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!} + \left[(m_2 - a_2) \right. \right.$$

$$\left. + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!} - (m_2 - a_2) \frac{\left[w - (a_1 + m_2 + b_3) \right]}{4!} \tag{50}$$

$$- (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} - \frac{\left[w - (a_1 + m_2 + b_3) \right]^5}{5!}$$

$$\left. - \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + a_2 + b_3) \right]^5}{5!} \right\}$$

$$h_2(w) = k_2 \left\{ \left[(m_1 - a_1)(m_3 - b_3) \right] \frac{\left[w - (m_1 + a_2 + m_3) \right]^3}{3!} + \left[(m_2 - a_2)(m_3 - b_3) \right] \right.$$

$$\frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!} + \left[(m_2 - a_2) + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!}$$

$$+ \left[(m_1 - a_1) + (m_3 - b_3) \right] \frac{\left[w - (m_1 + a_2 + m_3) \right]^4}{4!} - (m_2 - a_2) \frac{\left[w - (a_1 + m_2 + b_3) \right]^4}{4!}$$

$$- (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} - \frac{\left[w - (a_1 + m_2 + b_3) \right]^5}{5!} - \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!}$$

$$\left. + \frac{\left[w - (m_1 + a_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + a_2 + b_3) \right]^5}{5!} \right\} \tag{51}$$

$$
\begin{aligned}
h_2(w) = k_2 \Bigg\{ & \left[(m_1-a_1)(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^3}{3!} + \left[(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^3}{3!} \\
& + \left[(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^4}{4!} + \left[(m_1-a_1)+(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^4}{4!} \\
& -(m_1-a_1)\frac{\left[w-(m_1+a_2+b_3)\right]^4}{4!} (m_2-a_2)\frac{\left[w-(a_1+m_2+b_3)\right]^4}{4!} -(m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} \\
& -\frac{\left[w-(m_1+a_2+b_3)\right]^5}{5!} - \frac{\left[w-(a_1+m_2+b_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} \\
& + \frac{\left[w-(m_1+a_2+m_3)\right]^5}{5!} + \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} + \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!} \Bigg\}
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
h_2(w) = -k_2 \Bigg\{ & \left[(m_1-a_1)(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^2}{2!} + \\
& \left[(m_1-a_1)(m_2-a_2)+(m_2-a_2)(m_3-b_3)+(m_3-b_3)(m_1-a_1)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^3}{3!} - \\
& \left[(m_1-a_1)(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^3}{3!} - \left[(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^3}{3!} \\
& + \left[(m_1-a_1)+(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^4}{4!} + \left[(m_1-a_1)+(m_3-b_3)\right] \\
& \frac{\left[w-(m_1+a_2+m_3)\right]^4}{4!} + \left[(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^4}{4!} \\
& + (m_1-a_1)\frac{\left[w-(m_1+a_2+b_3)\right]^4}{4!} + (m_2-a_2)\frac{\left[w-(a_1+m_2+b_3)\right]^4}{4!} \\
& + (m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} + \frac{\left[w-(m_1+a_2+b_3)\right]^5}{5!} + \frac{\left[w-(a_1+m_2+b_3)\right]^5}{5!} \\
& + \frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} + \frac{\left[w-(m_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(m_1+a_2+m_3)\right]^5}{5!} \\
& - \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!} \Bigg\}
\end{aligned} \tag{53}
$$

$$h_2(w) = k_2 \left\{ \left[(m_1 - a_1)(m_3 - b_3) \right] \frac{\left[w - (m_1 + a_2 + m_3) \right]^3}{3!} \right.$$

$$+ \left[(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!} + \left[(m_1 - a_1) + (m_3 - b_3) \right]$$

$$\frac{\left[w - (m_1 + a_2 + m_3) \right]^4}{4!} + \left[(m_2 - a_2) + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!}$$

$$- (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} + \frac{\left[w - (m_1 + a_2 + m_3) \right]^5}{5!}$$

$$\left. - \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + a_2 + b_3) \right]^5}{5!} \right\} \tag{54}$$

$$h_2(w) = k_2 \left\{ \left[(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!} \right. \tag{55}$$

$$+ \left[(m_2 - a_2) + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!} - (m_3 - b_3)$$

$$\left. \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} - \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!} + \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} \right\}$$

$$h_2(w) = -k_2 \left\{ \left[(m_1 - a_1)(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (m_1 + m_2 + m_3) \right]^2}{2!} \right.$$

$$+ \left[(m_1 - a_1)(m_2 - a_2) + (m_2 - a_2)(m_3 - b_3) - (m_3 - b_3)(m_1 - a_1) \right] \frac{\left[w - (m_1 + m_2 + m_3) \right]^3}{3!}$$

$$- \left[(m_1 - a_1)(m_3 - b_3) \right] \frac{\left[w - (m_1 + a_2 + m_3) \right]^3}{3!} - \left[(m_2 - a_2)(m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^3}{3!}$$

$$+ \left[(m_1 - a_1) + (m_2 - a_2) + (m_3 - b_3) \right] \frac{\left[w - (m_1 + m_2 + m_3) \right]^4}{4!} + \frac{\left[w - (m_1 + m_2 + m_3) \right]^5}{5!}$$

$$- \left[(m_1 - a_1) + (m_3 - b_3) \right] \frac{\left[w - (m_1 + a_2 + m_3) \right]^4}{4!} - \left[(m_2 - a_2) + (m_3 - b_3) \right] \frac{\left[w - (a_1 + m_2 + m_3) \right]^4}{4!}$$

$$+ (m_2 - a_2) \frac{\left[w - (a_1 + m_2 + b_3) \right]^4}{4!} + (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3) \right]^4}{4!} + \frac{\left[w - (a_1 + a_2 + m_3) \right]^5}{5!}$$

$$\left. + \frac{\left[w - (a_1 + m_2 + b_3) \right]^5}{5!} - \frac{\left[w - (m_1 + a_2 + m_3) \right]^5}{5!} - \frac{\left[w - (a_1 + m_2 + m_3) \right]^5}{5!} - \frac{\left[w - (a_1 + a_2 + b_3) \right]^5}{5!} \right\} \tag{56}$$

$$h_2(w) = -k_2\Bigg\{\left[(m_1-a_1)(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^2}{2!}$$
$$+\left[(m_1-a_1)(m_2-a_2)+(m_2-a_2)(m_3-b_3)+(m_3-b_3)(m_1-a_1)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^3}{3!}$$
$$-\left[(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^3}{3!} - \left[(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^4}{4!}$$
$$+\left[(m_1-a_1)+(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^4}{4!} + (m_2-a_2)\frac{\left[w-(a_1+m_2+b_3)\right]}{4!}$$
$$+(m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} + \frac{\left[w-(a_1+m_2+b_3)\right]^5}{5!} + \frac{\left[w-(m_1+m_2+m_3)\right]^5}{5!}$$
$$+\frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\Bigg\} \quad (57)$$

$$h_2(w) = -k_2\Bigg\{\left[(m_1-a_1)(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^2}{2!}$$
$$+\left[(m_1-a_1)(m_2-a_2)+(m_2-a_2)(m_3-b_3)+(m_3-b_3)(m_1-a_1)\right]$$
$$\frac{\left[w-(m_1+m_2+m_3)\right]^3}{3!} - \left[(m_2-a_2)(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^3}{3!}$$
$$-\left[(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^4}{4!}$$
$$+\left[(m_1-a_1)+(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(m_1+m_2+m_3)\right]^4}{4!}$$
$$+(m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} + \frac{\left[w-(m_1+m_2+m_3)\right]^5}{5!}$$
$$+\frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\Bigg\} \quad (58)$$

$$h_2(w) = -k_2\left\{(m_3 - b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} + \frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\right\} \tag{59}$$

$$\begin{aligned} h_2(w) = -k_2\Bigg\{&\left[(m_1-a_1)(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^3}{3!} + (m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} \\ &-\left[(m_1-a_1)+(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^4}{4!} - \frac{\left[w-(m_1+a_2+m_3)\right]^5}{5!} \\ &-\frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\Bigg\} \end{aligned} \tag{60}$$

$$\begin{aligned} h_2(w) = -k_2\Bigg\{&-\left[(m_1-a_1)(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^3}{3!} + (m_1-a_1)\frac{\left[w-(m_1+a_2+b_3)\right]^4}{4!} \\ &-\left[(m_1-a_1)+(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^4}{4!} + (m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} \\ &+\frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(m_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\Bigg\} \end{aligned} \tag{61}$$

$$\begin{aligned} h_2(w) = -k_2\Bigg\{&\left[(m_1-a_1)(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^3}{3!} - \left[(m_2-a_2)(m_3-b_3)\right] \\ &\frac{\left[w-(a_1+m_2+m_3)\right]^3}{3!} - \left[(m_1-a_1)+(m_3-b_3)\right]\frac{\left[w-(m_1+a_2+m_3)\right]^4}{4!} \\ &-\left[(m_2-a_2)+(m_3-b_3)\right]\frac{\left[w-(a_1+m_2+m_3)\right]^4}{4!} + (m_1-a_1)\frac{\left[w-(m_1+a_2+b_3)\right]^4}{4!} \\ &+(m_3-b_3)\frac{\left[w-(a_1+a_2+m_3)\right]^4}{4!} + \frac{\left[w-(a_1+a_2+m_3)\right]^5}{5!} + \frac{\left[w-(m_1+a_2+b_3)\right]^5}{5!} \\ &-\frac{\left[w-(m_1+a_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+m_2+m_3)\right]^5}{5!} - \frac{\left[w-(a_1+a_2+b_3)\right]^5}{5!}\Bigg\} \end{aligned} \tag{62}$$

$$h_2(w) = k_2 \left\{ -\left[(m_1 - a_1)(m_3 - b_3)\right] \frac{\left[w - (m_1 + a_2 + m_3)\right]^3}{3!} + \left[(m_1 - a_1) + (m_3 - b_3)\right] \frac{\left[w - (m_1 + a_2 + m_3)\right]^4}{4!} - (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3)\right]^4}{4!} + \frac{\left[w - (m_1 + a_2 + m_3)\right]^5}{5!} - \frac{\left[w - (a_1 + a_2 + m_3)\right]^5}{5!} \right\} \tag{63}$$

$$\begin{aligned} h_2(w) = -k_2 \Bigg\{ & \left[(m_1 - a_1)(m_2 - a_2)(m_3 - b_3)\right] \frac{\left[w - (m_1 + m_2 + m_3)\right]^2}{2!} - \frac{\left[w - (a_1 + m_2 + m_3)\right]^5}{5!} \\ & + \left[(m_1 - a_1)(m_2 - a_2) + (m_2 - a_2)(m_3 - b_3) - (m_3 - b_3)(m_1 - a_1)\right] \frac{\left[w - (m_1 + m_2 + m_3)\right]^3}{3!} \\ & - \left[(m_1 - a_1)(m_3 - b_3)\right] \frac{\left[w - (m_1 + a_2 + m_3)\right]^3}{3!} - \left[(m_2 - a_2)(m_3 - b_3)\right] \frac{\left[w - (a_1 + m_2 + m_3)\right]^3}{3!} \\ & + \left[(m_1 - a_1) + (m_2 - a_2) + (m_3 - b_3)\right] \frac{\left[w - (m_1 + m_2 + m_3)\right]^4}{4!} - \left[(m_1 - a_1) + (m_3 - b_3)\right] \\ & \frac{\left[w - (m_1 + a_2 + m_3)\right]^4}{4!} - \left[(m_2 - a_2) + (m_3 - b_3)\right] \frac{\left[w - (a_1 + m_2 + m_3)\right]^4}{4!} + (m_1 - a_1) \\ & \frac{\left[w - (m_1 + a_2 + b_3)\right]^4}{4!} + (m_3 - b_3) \frac{\left[w - (a_1 + a_2 + m_3)\right]^4}{4!} + \frac{\left[w - (a_1 + a_2 + m_3)\right]^5}{5!} \\ & + \frac{\left[w - (m_1 + m_2 + m_3)\right]^5}{5!} + \frac{\left[w - (m_1 + a_2 + b_3)\right]^5}{5!} - \frac{\left[w - (m_1 + a_2 + m_3)\right]^5}{5!} - \frac{\left[w - (a_1 + a_2 + b_3)\right]^5}{5!} \Bigg\} \end{aligned} \tag{64}$$

Table 1. Sequence of points considered to determine the sub cases of Case II [20,21]

Sub Case	**Sequence of points determining intervals for *w***	**Equations of Giving pdf for respective interval**
(i)	$a_1 + a_2 + m_3 < a_1 + a_2 + b_3 < a_1 + m_2 + m_3 < a_1 + m_2 + b_3 <$ $m_1 + a_2 + m_3 < m_1 + a_2 + b_3 < m_1 + m_2 + m_3 < m_1 + m_2 + b_3$	(5.1), (5.2), (5.3), (5.4),(5.5),(5.6), (5.7),
(ii)	$a_1 + a_2 + m_3 < a_1 + a_2 + b_3 < a_1 + m_2 + m_3 < a_1 + m_2 + b_3 =$ $m_1 + a_2 + m_3 < m_1 + a_2 + b_3 < m_1 + m_2 + m_3 < m_1 + m_2 + b_3$	(5.1), (5.2), (5.3), (5.4), (5.6), (5.7),
(iii)	$a_1 + a_2 + m_3 < a_1 + a_2 + b_3 < a_1 + m_2 + m_3 < m_1 + a_2 + m_3$ $< a_1 + m_2 + b_3 < m_1 + a_2 + b_3 < m_1 + m_2 + m_3 < m_1 + m_2 + b_3$	(5.1), (5.2), (5.3), (5.8), (5.6), (5.7),
(iv)	$a_1 + a_2 + m_3 < a_1 + a_2 + b_3 = a_1 + m_2 + m_3 < a_1 + m_2 + b_3$ $< m_1 + a_2 + m_3 < m_1 + a_2 + b_3 = m_1 + m_2 + m_3 < m_1 + m_2 + b_3$	(5.1), (5.3), (5.4), (5.5), (5.7),

Sub Case	**Sequence of points determining intervals for *w***	**Equations of Giving pdf for respective interval**
(v)	$a_1+a_2+m_3<a_1+a_2+b_3=a_1+m_2+m_3<a_1+m_2+b_3=$ $m_1+a_2+m_3<m_1+a_2+b_3=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.3), (5.5), (5.7),
(vi)	$a_1+a_2+m_3<a_1+a_2+b_3=a_1+m_2+m_3<m_1+a_2+m_3<$ $a_1+m_2+b_3<m_1+a_2+b_3=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.3), (5.5), (5.7)
(vii)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<a_1+m_2+b_3<$ $m_1+a_2+m_3<m_1+m_2+m_3<m_1+a_2+b_3<m_1+m_2+b_3$	(5.1), (5.9), (5.3), (5.4), (5.5), (5.10), (5.7),
(viii)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<a_1+m_2+b_3=$ $m_1+a_2+m_3<m_1+m_2+m_3<m_1+a_2+b_3<m_1+m_2+b_3$	(5.1), (5.9), (5.3), (5.5), (5.10), (5.7)
(ix)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<m_1+a_2+m_3<$ $a_1+m_2+b_3<m_1+m_2+m_3<m_1+a_2+b_3<m_1+m_2+b_3$	(5.1), (5.9), (5.3), (5.8), (5.5), (5.10), (5.7),
(x)	$a_1+a_2+b_3<a_1+a_2+b_3<a_1+m_2+m_3<a_1+m_2+b_3$ $<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.2), (5.3), (5.4), (5.11),
(xi)	$a_1+a_2+m_3<a_1+a_2+b_3<a_1+m_2+m_3<a_1+m_2+b_3$ $=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.2), (5.3), (5.4), (5.11),
(xii)	$a_1+a_2+m_3<a_1+a_2+b_3<a_1+m_2+m_3<m_1+m_2+m_3$ $<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1), (5.3), (5.4), (5.12), (5.11),
(xiii)	$a_1+a_2+m_3<a_1+a_2+b_3=a_1+m_2+m_3<a_1+m_2+b_3$ $<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.3), (5.4), (5.11),
(xiv)	$a_1+a_2+m_3<a_1+a_2+b_3=a_1+m_2+m_3<a_1+m_2+b_3$ $=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.9), (5.11),
(xv)	$a_1+a_2+m_3<a_1+a_2+b_3=a_1+m_2+m_3<m_1+m_2+m_3$ $<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1), (5.9), (5.12), (5.11),
(xvi)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<a_1+m_2+b_3$ $<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.13), (5.3), (5.5), (5.11),
(xvii)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<a_1+m_2+b_3$ $=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.13), (5.3), (5.11),
(xviii)	$a_1+a_2+m_3<a_1+m_2+m_3<a_1+a_2+b_3<m_1+m_2+m_3$ $<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1), (5.13), (5.3), (5.12), (5.11),
(xix)	$a_1+a_2+m_3<a_1+a_2+b_3<m_1+a_2+m_3<m_1+a_2+b_3<$ $a_1+m_2+m_3<a_1+m_2+b_3<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.9), (5.14), (5.16), (5.6), (5.7)

Sub Case	Sequence of points determining intervals for *w*	Equations of Giving pdf for respective interval
(xx)	$a_1+a_2+m_3<a_1+a_2+b_3<m_1+a_2+m_3, m_1+a_2+b_3=$ $a_1+m_2+m_3<a_1+m_2+b_3<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.9), (5.14), (5.16), (5.6), (5.7),
(xxi)	$a_1+a_2+m_3<a_1+a_2+b_3<m_1+a_2+m_3<a_1+m_2+m_3<$ $m_1+a_2+b_3<a_1+m_2+b_3<m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.2), (5.14), (5.8), (5.16), (5.6), (5.7),
(xxii)	$a_1+a_2+m_3<a_1+a_2+b_3=m_1+a_2+m_3<m_1+a_2+b_3<$ $a_1+m_2+m_3<a_1+m_2+b_3=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.14), (5.15) (5.16), (5.7),
(xxiii)	$a_1+a_2+m_3<a_1+a_2+b_3=m_1+a_2+m_3<m_1+a_2+b_3=$ $a_1+m_2+m_3<a_1+m_2+b_3=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1), (5.14), (5.16), (5.7),
(xxiv)	$a_1+a_2+m_3<a_1+a_2+b_3=m_1+a_2+m_3<a_1+m_2+m_3<$ $m_1+a_2+b_3<a_1+m_2+b_3=m_1+m_2+m_3<m_1+m_2+b_3$	(5.1),c(5.14),c(5.8), (5.16),c(5.7),
(xxv)	$a_1+a_2+m_3<m_1+a_2+m_3<a_1+a_2+b_3<m_1+a_2+b_3<$ $a_1+m_2+m_3<m_1+m_2+m_3<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1),c(5.17),c(5.14), (5.15), c(5.16), (5.18), (5.7)
(xxvi)	$a_1+a_2+m_3<m_1+a_2+m_3<a_1+a_2+b_3<m_1+a_2+b_3=$ $a_1+m_2+m_3<m_1+m_2+m_3<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1), (5.17), (5.14), (5.16), (5.18), (5.7)
(xxvii)	$a_1+a_2+m_3<m_1+a_2+m_3<a_1+a_2+b_3<a_1+m_2+m_3$ $<m_1+a_2+b_3<m_1+m_2+m_3<a_1+m_2+b_3<m_1+m_2+b_3$	(5.1), (5.17), (5.14), (5.8), (5.18), (5.7)

Table 2. Changes involved in the discussion of Cases III to VIII & I

Case	Pdf	Equations (3.1) to (3.8)	Sub cases (i) to (xxvii)
III	$h_3(w)$	$a_2\to b_2, b_3\to a_3, k_2\to k_3$	$a_2\to m_2, m_2\to b_2, b_3\to m_3, m_3\to a_3$
IV	$h_4(w)$	$a_2\to b_2, k_2\to k_4$	$a_2\to m_2, m_2\to b_2$
V	$h_5(w)$	$a_1\to b_1, b_3\to a_3, k_2\to k_5$	$a_1\to m_1, m_1\to b_1, b_3\to m_3, m_3\to a_3$
VI	$h_6(w)$	$a_1\to b_1, k_2\to k_6$	$a_1\to m_1, m_1\to b_1$
VII	$h_7(w)$	$a_1\to b_1, a_2\to b_2, b_3\to a_3,$ $k_2\to k_7$	$a_1\to m_1, m_1\to b_1, a_2\to m_2, m_2\to b_2,$ $b_3\to m_3,\ m_3\to a_3$
VIII	$h_8(w)$	$a_1\to b_1, a_2\to b_2, k_2\to k_8$	$a_1\to m_1, m_1\to b_1, a_2\to m_2, m_2\to b_2$
I	$h_1(w)$	$b_3\to a_3, k_2\to k_1$	$b_3\to m_3, m_3\to a_3$

where

$$k_2 = \frac{8}{(b_1 - a_1)(b_2 - a_2)(b_3 - a_3)(m_1 - a_1)(m_2 - a_2)(b_3 - m_3)},$$

$$k_3 = \frac{(m_2 - a_2)(b_3 - m_3)}{(b_2 - m_2)(m_3 - a_3)} k_2,$$

$$k_4 = \frac{(m_2 - a_2)}{(b_2 - m_2)} k_2,$$

$$k_5 = \frac{(m_1 - a_1)(b_3 - m_3)}{(b_1 - m_1)(m_3 - a_3)} k_2,$$

$$k_6 = \frac{(m_1 - a_1)}{(b_1 - m_1)} k_2,$$

$$k_7 = \frac{(m_1 - a_1)(m_2 - a_2)(b_3 - m_3)}{(b_1 - m_1)(b_2 - m_2)(m_3 - a_3)} k_2,$$

$$k_8 = \frac{(m_1 - a_1)(m_2 - a_2)}{(b_1 - m_1)(b_2 - m_2)} k_2,$$

and $$k_1 = \frac{(b_3 - m_3)}{(m_3 - a_3)} k_2.$$

6. Generalization for n - Variables

Here, we have discussed the generalization of random variables to 'n' by considering the total number of cases along with its subcases.

(A) For the total number of cases considered as sum of two triangular random variables are 2^2, as sum of three triangular random variables are 2^3, in the similar manner if there are sum of *n*–triangular random variables then there are 2^n cases.

We see that the equation (3.1) plays major role in the discussion of Case II, therefore the generalization for n -number of random variables is as follows ($n \geq 2$)

$$h_2(w) = (-)^n K_2 \Bigg[\left(A_1 A_2 A_3,\ldots,A_{n-1}B_n\right)\frac{W^{n-1}}{(n-1)!} + \Big\{\left(A_1A_2A_3,\ldots,A_{n-1}\right) +$$

$$\left(A_2A_3,\ldots,A_{n-1}B_n\right) +,\ldots,+ \left(B_nA_{n-2},\ldots,A_1A_2\right)\Big\}\frac{W^n}{(n)!} + \qquad (65)$$

$$\Big\{\left(A_1A_2A_3,\ldots,A_{n-2}\right) + \left(A_2A_3,\ldots,A_{n-1}\right) + \left(B_nA_{n-3},\ldots,A_2A_1\right)\Big\}\frac{W^{n+1}}{(n+1)!}$$

$$+,\ldots,+\left(A_1 + A_2 + A_3 +,\ldots,+A_{n-1} + B_n\right)\frac{W^{2n-2}}{(2n-2)!} + \frac{W^{2n-1}}{(2n-1)!}\Bigg], \qquad \ldots(6.1)$$

where

$$W = w - \left(m_1 + m_2 + \ldots\ldots m_n\right), \qquad (66)$$

$$A_k = m_k - a_k \text{, } (k=1, 2, 3,\ldots, n-1) \text{ and } B_n = m_n - b_n \text{ .} \qquad (67)$$

Other 2^n-1 equations corresponding to the equations from (3.2) to (3.8) of the Case II will be calculated for n - random variables in view of the Remark

(B) Regarding sub cases

When number of random variables is two, we have three sub cases $\left(a_1 + m_2 < m_1 + a_2,\, a_1 + m_2 = m_1 + a_2,\, a_1 + m_2 > m_1 + a_2\right)$ for each case.

When number of random variables is three we find that each of above sub case give rise to 3^2 cases. Thus total number of sub cases becomes 3^1 x 3^2 = 3^3 =27.

If we further observe the pattern we find that in case of four variables each of the above sub case will give rise to 3^3 cases. So the total no. of sub cases are 3^1 x 3^2 x 3^3 = 3^6.

Similarly in case of n random variables then number of sub cases

$=3^1$ x 3^2 x 3^3x……x3^{n-1} = $3^{\frac{(n-1)n}{2}}$

For four variables, X, Y, Z and U having the supports $a_1 \le x \le b_1$, $a_2 \le y \le b_2$, $a_3 \le z \le b_3$, $m_4 \le u \le b_4$ with modes m_1, m_2, m_3 and m_4 respectively, the generalization of equation (A) as follows

$$\left.\begin{array}{l} a_1 + a_2 + a_3 + m_4 < w < a_1 + a_2 + a_3 + b_4 \\ a_1 + a_2 + m_3 + m_4 < w < a_1 + a_2 + m_3 + b_4 \\ a_1 + m_2 + a_3 + m_4 < w < a_1 + m_2 + a_3 + b_4 \\ a_1 + m_2 + m_3 + m_4 < w < a_1 + m_2 + m_3 + b_4 \\ m_1 + a_2 + a_3 + m_4 < w < m_1 + a_2 + a_3 + b_4 \\ m_1 + a_2 + m_3 + m_4 < w < m_1 + a_2 + m_3 + b_4 \\ m_1 + m_2 + a_3 + m_4 < w < m_1 + m_2 + a_3 + b_4 \\ m_1 + m_2 + m_3 + m_4 < w < m_1 + m_2 + m_3 + b_4 \end{array}\right\}$$

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