HYPERSURFACES OF MATSUMOTO CONFORMAL CHANGED FINSLER SPACES

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Abstract: In the year 1984, Shibata et al. [16] investigated the theory of a change which has been called a $\beta$-change of a Finsler metric. On the other hand, in 1985 a systematic study of geometry of hypersurfaces of Finsler spaces was given by Matsumoto.[10] The present paper is devoted to study the condition for a Matsumoto conformal change to be projective and find out when a totally geodesic hypersurface $F^{*}$ remains to be a totally geodesic hypersurface $F^*$ under the projective Matsumoto conformal change.[1] Further, we obtained the condition under which a Finslerian hypersurface given by the projective Matsumoto conformal change are projectively flat.

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1. Introduction
Let $(M^n, L)$ be an n-dimensional Finsler space on a differential manifold $M^n$, equipped with the fundamental function $L(x, y)$. In 1984, Shibata et.al. [16] introduced the transformation of Finsler metric as

$$L(x, y) = f(L, \beta).$$

where $\beta = b_i(x)y^i$ and $b_i(x)$ is a covariant vector in $(M^n, L)$. Here $f$ is positively homogeneous function of degree one in $L$ and $\beta$. This change of metric has been called $\beta$-change.[8]

The conformal theory of a Finsler spaces has been initiated by Knebelman[9], in 1929 and has been investigated in detail by many authors in the papers ([4],[5],[6],[13],[17],[18] etc.)

The conformal change has been defined as

$$L(x, y) \rightarrow e^{\sigma(x)}L(x, y)$$
where $\sigma(x)$ is a function of position only and known as conformal factor.

In the present paper, we have studied a transformation which combines above two transformations with Matsumoto metric, which generalizes all the above transformations. In fact, we consider a change of the form

$$L(x, y) \to \ast L(x, y) = e^{\sigma(x)} \frac{L^2(x, y)}{L(x, y) - \beta(x, y)},$$

(1)

where $\sigma(x)$ is a function of position only and $\beta(x, y) = b_i(x)y^i$ is 1-form in $M^n$, which we shall call a Matsumoto conformal change. This change generalizes various types of change. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to Matsumoto change. When $\beta = 0$ and $\sigma$ is a non-zero constant then it reduces to homothetic change.

In this paper, we have obtained the condition for a Matsumoto conformal change to be projective and find out when a totally geodesic hypersurface $\ast F_n^{n-1}$ remains to be a totally geodesic hypersurface $F_n^{n-1}$ under the projective Matsumoto conformal change. Further, we have obtained the condition under which a Finslerian hypersurfaces given by the projective conformal change are projectively flat.

Throughout the paper we shall confine ourselves to Cartan’s connection, and the notations and terminology of the monograph [11] will be used without comment. In the paper monograph of Matsumoto[11] will be quoted by (#).

2. Relation among the quantities of Finsler spaces $F^n$ and $\ast F^n$

Let $(M^n, L)$ be a Finsler space $F^n$, where $M^n$ is an n-dimensional differentiable manifold equipped with a fundamental function $L$, defined by equation (1) has been called Matsumoto conformal change, where $\sigma(x)$ is conformal factor which is function of position only and $\beta(x, y) = b_i(x)y^i$ is 1-form in $M^n$. A space equipped with fundamental metric $\ast L(x,y)$ is called conformally changed space $\ast F^n$.

Differentiating equation (1) with respect to $y^i$, we obtain the relation among normalized supporting element $l_i$ and $\ast l_i$:

$$\ast L_i = \ast l_i = \tau(l_i + e^{\sigma}m_i)$$

(2)

Throughout the paper, we shall use the notations, symbols and well known relations used in the paper [10].

$$L_i = \frac{\partial L}{\partial y^i} = l_i, \quad L_{ij} = \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad \tau = \frac{\ast L}{L} = e^{\sigma(x)} \frac{L}{L - \beta}, \quad \text{and}$$
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\[ m_i = (e^\sigma - \tau)l_i + i \cdot \delta. \]  \hspace{1cm} (3)

The vector \( m_i \) is orthogonal to the normalize supporting element \( l_i \).

Again, differentiating equation (2) with respect to \( y^j \) we obtain

\[ ^*L_{ij} = \frac{\tau}{L} \left( (2 - e^{-\tau})h_{ij} + 2e^{-2\sigma}m_im_j \right) \]  \hspace{1cm} (4)

where

\[ \frac{\partial \tau}{\partial y^j} = \frac{m_j}{L - \beta} = e^{-\sigma} \frac{\tau}{L} m_j \quad \text{and} \quad \frac{\partial m_j}{\partial y^j} = \frac{1}{L} \left( (e^{-\sigma} - 1)h_{ij} + (e^{-\sigma}m_i - l_i)m_j \right). \]

If we denote \( g_{ij}(x, y) \) the fundamental tensor \( \frac{\partial^2 L^2}{2\partial y^i \partial y^j} \) and put

\[ h_{ij} = g_{ij} - l_i l_j. \]  \hspace{1cm} (5)

Then in virtue of (5) and (2) the equation (4) is rewritten as the relation between the fundamental tensors

\[ ^*g_{ij} = \tau^2 \left( (2 - e^{-\sigma})g_{ij} + (e^{-\sigma} - 1)l_i l_j + 3e^{-2\sigma}m_im_j + e^{-\sigma}(m_i - l_i)m_j \right). \]  \hspace{1cm} (6)

Reciprocal of \( ^*g^{ij} \) of \( ^*g_{ij} \) has been worked out using the fact that \( ^*g^{jk} \cdot ^*g_{ij} = \delta^i_k. \)

\[ ^*g^{ij} = \frac{1}{\tau^2 \mu} \left( Ag^{ij} - B(l^i m^j + m^i l^j) - 2e^{-\sigma}m^i m^j - Cl^i l^j \right). \]  \hspace{1cm} (7)

where \( \mu = AB, A = 2m^2e^{-\sigma} + 2e^\sigma - \tau, B = 2 - e^{-\sigma}, \) and

\[ C = (e^{-\sigma} - 1)(A + m^2e^{-\sigma}) \]

Next we obtain easily the relation between \( ^*L_{ijk} \) and \( L_{ijk} \) by differentiating equation (5) with respect to \( y^k \) and using the relation

\[ \frac{\partial h_{ij}}{\partial y^k} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} + l_j h_{ik}), \]  \hspace{1cm} we obtain

\[ ^*L_{ijk} = \frac{2\tau}{L^2} \left( (2 - e^{-\sigma})C_{ijk} + e^{-2\sigma} \left( e^\sigma - \tau \right) e_{ijk} h_{ij} m_k \right) \]

\[ - \frac{1}{2} e_{ijk} (2 - e^{-\sigma})h_{ij} l_k + e^{-2\sigma} e_{ijk} l_i m_j m_k. \]
+ 3e^{-2\sigma} m_j m_k \}\right) \tag{8}

where the symbol $\Theta_{ijk}$ denote the sum of cyclic permutation of indices i, j and k. Meaning there by

$$\Theta_{ijk} m_k = h_i m_k + h_j m_i + h_k m_j$$

From the equation (5) we have

$$\frac{\partial h_{ij}}{\partial y^k} = 2C_{ijk} - \frac{1}{L} (l_i h_{jk} + l_j h_{ik}), \tag{9}$$

where $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}$ and $\frac{\partial l_i}{\partial y^j} = \frac{h_{ij}}{L}$. Also, from above equation we get

$$L_{ij} = \frac{2}{L} C_{ijk} - \frac{1}{L} (h_{ij} l_k + h_{jk} l_i + h_{ik} l_j) \tag{10}$$

In the virtue of equation (9) we obtain the relation between $C_{ijk}$ and $^{*}C_{ijk}$ from (8)

$$^{*}C_{ijk} = \frac{\tau^2}{L} \left\{ LBC_{ijk} + (4 - 3\tau) \left( h_{ij} m_k + h_{jk} m_i + h_{ik} m_j \right) + 12m_j m_k \right\} \tag{11}$$

Now, we consider the relation between the Cartan's connection $F_{jk}^i$ and $^{*}F_{jk}^i$ in the paper as [11]

$$D_j^i = ^{*}F_{jk}^i - F_{jk}^i \tag{12}$$

Transvecting above equation by $y^j$ and using $F_{jk}^i y^j = G_k^i$, and $D_j^i y^j = D_k^i$, we get

$$^{*}G_k^i = G_k^i + D_k^i \tag{13}$$

where the subscript '0' denote the contraction by the supporting element $y^j$.

Further, transvecting (13)by $y^k$ and using $G_k^i y^k = G^i$, and $D_k^i y^k = D^i$, we get

$$^{*}G^i = G^i + D^i \tag{14}$$

These relations used in latter articles.
3. Relation between projective change and Matsumoto conformal change

For two Finsler spaces \( F^n = (M^n, L) \) and \( *F^n = (M^n, *L) \) if any geodesic of \( F^n \) is also a geodesic of \( *F^n \) and vice-versa, then the change \( L \rightarrow *L \) is called projective change. A geodesic of \( F^n \) is given by a system of differential equations

\[
\frac{d^2 y^i}{dt^2} + 2G^i(x, y) = y^i, \quad \text{where} \quad y^i = \frac{dx^i}{dt},
\]

and \( G^i(x, y) \) are second degree positively homogeneous function in \( y^i \). We are now in a position to find condition under which Matsumoto conformal change is projective. Let us consider Euler’s-Lagrange differential equation of a geodesic in the form,

\[
B_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) = 0.
\]

Similar Euler-Lagrange’s differential equation for \( *L \), will be

\[
* B_i = \frac{\partial *L}{\partial x^i} - \frac{d}{dt} \left( * L_i \right) = 0. \tag{15}
\]

Differentiating \( *L = \tau L \) with respect to \( x^i \), we have

\[
\frac{\partial *L}{\partial x^i} = \tau \left\{ (2 - \tau \sigma) \frac{\partial L}{\partial x^i} + L \frac{\partial \sigma}{\partial x^i} + \tau \sigma \frac{\partial \beta}{\partial x^i} \right\}.
\]

Differentiating equation (2) with respect to \( t \), we have

\[
\frac{d}{dt} \left( L_i \right) = \tau (2 - \tau \sigma) \frac{dL}{dt} + \left\{ (1 - \tau \sigma) y_i + \tau \theta_i \right\} \frac{d\tau}{dt}.
\]

Substituting above values in equation (15), we get

\[
* B_i = \tau (2 - \tau \sigma) B_i + \tau^2 \rho^{-\sigma} \frac{\partial \beta}{\partial x^i} - \left\{ (1 - \tau \sigma) y_i + \tau \theta_i \right\} \frac{d\tau}{dt} = 0. \tag{16}
\]

which leads to

\[
* B_i = e^{\sigma(x)} B_i + A_i = 0. \tag{17}
\]

where, is a covariant vector.

Thus we have,

**Theorem 3.1** Matsumoto conformal change (1) is projective iff covariant vector \( A_i \), given by equation (17) vanishes identically.
4. Hypersurface of a projective Matsumoto conformal changed space

A hypersurface \( M^{n-1} \) of a underlying smooth manifold \( M^n \) may be parametrically represented by the equation \( x^i = x^i(u^a) \), where \( u^a \) are Gaussian co-ordinate on \( M^{n-1} \) [3] and \( \alpha \) varies from 1 to \( n-1 \). Here we shall assume that the matrix consisting of the projection factor \( B^i_a = \frac{\partial x^i}{\partial u^a} \) is of rank \( n-1 \).

To introduce a Finsler structure in \( M^{n-1} \), the supporting element \( y^i \) at a point \((u^a)\) of \( M^{n-1} \) is assumed to be tangential to \( M^{n-1} \), so that we may write

\[
y^i = B^i_a(u)v^a.
\]

Thus \( v^a \) is thought of as supporting element of \( M^{n-1} \) at the point \((u^a)\).

Since the function \( L(u,v) = L(x(u),y(u,v)) \) gives rise to a Finsler metric of \( M^{n-1} \), we get a \((n-1)\)dimensional Finsler space \( F^{n-1} = \{M^{n-1}, L(u,v)\} \).

The fundamental function \( L(u,v) \) of this Finslerian hypersurfaces \( F^{n-1} \) of \( F^n \) is called the induced metric of \( F^{n-1} \).

\[
B^i_{a\beta} = \frac{\partial B^i_a}{\partial u^\beta} = \frac{\partial x^i}{\partial u^a \partial u^\beta}, \quad B^i_{0\beta} = v^a B^i_{a\beta}.
\]

At each point \((u^a)\) of \( F^{n-1} \), the unit normal vector \( N^i(u,v) \) is defined by

\[
(i) \quad g_{ij}B^j_a N^j = 0, \quad (ii) \quad g_{ij}N^i N^j = 1.
\]

If \((B^i_a, N^i)\) is the inverse of the matrix \((B^i_a, N^i)\), then

\[
B^i_a B^\beta_i = \delta^a_\beta, \quad B^i_a N^j_i = 0, \quad N^i N^j_i = 1 \quad \text{and} \quad B^i_a B^\gamma_j + N^i N^j_i = \delta^i_j.
\]

\[\therefore \quad B^i_a = g^{a\beta} g_{ij}, \quad N^i = g_{ij} N^j.\]

For induced Cartan's connection \( CT = (F^a_{\beta \gamma}, N^i_\alpha, C^a_{\beta \gamma}) \) on \( F^{n-1} \), the normal curvature vector \( H^i_\alpha \) is given by,

\[
H^i_\alpha = N_i (B^i_{0\alpha} + N^i_j B^j_\alpha).
\]

Consider a Finslerian hypersurface \( F^{n-1} = \{M^{n-1}, L(u,v)\} \) of the \( F^n \) and another Finsler hypersurface \( ^* F^{n-1} = \{M^{n-1}, L(u,v)\} \) of the \(^* F^n \) given by the Matsumoto
conformal change. Let $N^i$ be the unit vector at each point of $F^{n-1}$ and $(B_\alpha^i, N_i)$ be the inverse of the matrix of $(B_\alpha^i, N_i)$. The function $B_\alpha^i$ may be considered as component of (n-1) linearly independent tangent vectors of $F^{n-1}$ and they are invariant under Matsumoto conformal change. Thus we shall show that a unit normal vector $^*N^i(u,v)$ of $F^{n-1}$ is uniquely determined by

\begin{align}
(i) \quad & ^*g_{ij}B_\alpha^iN^j = 0, \\
(ii) \quad & ^*g_{ij}N^iN^j = 1
\end{align}

(23)

Contracting (6) by $N^iN^j$ and paying attention to the fact that $l_iN^i = 0$ and using equation (20) we have

\[ ^*g_{ij}N^iN^j = \tau^2 \|2 - \tau e^{-\sigma}\| + 3e^{-2\sigma}\tau^2(b_iN^i)^2 \]

It can also be written as

\[ ^*g_{ij} \left\{ \pm \frac{N_i}{\tau\sqrt{(2 - \tau e^{-\sigma}) + 3e^{-2\sigma}\tau^2(b_iN^i)^2}} \right\} \times \left\{ \pm \frac{N^j}{\tau\sqrt{(2 - \tau e^{-\sigma}) + 3e^{-2\sigma}\tau^2(b_iN^i)^2}} \right\} = 1. \]

(24)

Since,

\[ ^*N^i = \frac{N^i}{\tau\sqrt{(2 - \tau e^{-\sigma}) + 3e^{-2\sigma}\tau^2(b_iN^i)^2}}. \]

(25)

Substituting the values from equations (16) and (25) in the equation (i) of (23) we have

\[ \tau^2 \left\{ \|2 - \tau e^{-\sigma}\|g_{ij} + (e^{-\sigma} - 1)l_j + 3m_i m_j e^{-2\sigma} + \tau e^{-\sigma}(l_i m_j + l_j m_i) \right\} \times \frac{B_\alpha^iN^j}{\tau\sqrt{(2 - \tau e^{-\sigma}) + 3e^{-2\sigma}\tau^2(b_iN^i)^2}} = 0. \]

Contracting above equations by $v^\alpha$ and using $y^i = B_\alpha^iv^\alpha$, we get

\[ \tau e^{-\sigma} \left\{ 3\tau e^{-\sigma}(b_i - l_i) + 4l_i B_\alpha^i \right\} = 0. \]

In the virtue of (20) above equation becomes
\[
\left\{ 3\pi^2 e^{-2\sigma} (b_i - l_i) + 4\pi e^{-\sigma} l_i \right\} \frac{B_i^j N^j b_j}{\tau \sqrt{(2 - \pi e^{-\sigma}) + 3\pi e^{-2\sigma} \tau^2 (b_j N^j)^2}}
\]

(26)

\[
= 0.
\]

which implies that

\[
3\pi e^{-\sigma} (\beta - L) + 4L = 0.
\]

Thus, we have

\[
L = 0 \quad \left( \therefore \pi e^{-\sigma} (L - \beta) = L \right),
\]

which contradicts the assumption that \( L > 0 \). So that (25) gives \( b_i N^j = 0 \). Therefore equation (25) can be rewritten as

\[
^* N^i = \frac{N^i}{\tau \sqrt{(2 - \pi e^{-\sigma})}}
\]

(27)

Thus we have

**Proposition 4.1** If \( \{ (B_{i}^{\alpha} N^{i}), \alpha = 1, 2, 3, \ldots, (n - 1) \} \) be the field of linear frame of the Finsler space \( \mathcal{F}^n \) and we consider \( \{ (B_{i}^{\alpha} N^{i}), \alpha = 1, 2, 3, \ldots, (n - 1) \} \) as a linear frame of the Finsler space \( \mathcal{F}^n \) such that (4.10) holds good along \( \mathcal{F}^{n-1} \) then \( b_i \) is tangential to both the hypersurfaces \( \mathcal{F}^{n-1} \) and \( \mathcal{F}^{n-1} \).

Now, we are going to examine, under what condition Matsumoto conformal change of the metric is projective also.

The quantities \( ^* B_{i}^{\alpha} \) are uniquely has been defined [10] along \( \mathcal{F}^{n-1} \) by

\[
^* B_{i}^{\alpha} = ^* g^{\alpha \beta} g_{ij} B_{\beta}^j
\]

where \( ^* g^{\alpha \beta} \) is the inverse matrix of \( ^* g_{\alpha \beta} \).

Let \( \{ ^* B_{i}^{\alpha}, ^* N_{j} \} \) be the inverse matrix of \( \{ B_{i}^{\alpha}, N^{j} \} \), then

\[
B_{\alpha}^i \cdot ^* B_{i}^{\beta} = \delta_{\alpha}^{\beta}, \quad B_{\alpha}^i \cdot ^* N_{j} = 0, \quad ^* N^{i} \cdot N_{i} = 1.
\]

In the virtue of \( B_{\alpha}^i \cdot ^* B_{j}^{\alpha} + ^* N^{i} \cdot N_{j} = \delta_{ij} \) and \( ^* N_{i} = ^* g_{ij} N^{j} \) the equations (6) and (27) give
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\[ *N_i = ^*g_{ij} *N^j = \frac{N^j}{\tau \sqrt{2 - \tau e^{-\alpha}}} = \tau \sqrt{2 - \tau e^{-\alpha}} N_i, \]

\[ *N_i = \tau \sqrt{2 - \tau e^{-\alpha}} N_i. \]  

(28)

Differentiating equation (14) with respect to \( y^j \) and using \( D^j_j = \frac{\partial D^j}{\partial y^j} \) and \( N^j_j = \frac{\partial G^j}{\partial y^j} \), we get

\[ D^j_j = ^*N^j_j - N^j_j. \]  

(29)

Further, contracting above equation by \( N_i B^j_a \) we have

\[ N_i D^j_j B^j_a = ^*N^j_j N_i B^j_a - N^j_j N_i B^j_a, \]

\[ \Rightarrow N_i D^j_j B^j_a = 0. \]  

(30)

If each geodesic of \( F^{(n-1)} \) with respect to the induced metric is also a geodesic of \( F^n \), then \( F^{(n-1)} \) is called totally geodesic[10]. A totally geodesic hypersurfaces is characterized by \([2,7,12,14]\) \( H_a = 0 \).

From (22), (28) and (30) we have

\[ ^*H_a = ^*N_i (^*B^j_a + ^*N^j_j ^*B^j_a), \]

\[ = ^*N_i B^j_a + ^*N_i D^j_j ^*B^j_a + ^*N_i N^j_j ^*B^j_a, \]

\[ ^*H_a = \tau \sqrt{2 - \tau e^{-\alpha}} H_a + N_i D^j_j ^*B^j_a. \]  

(31)

In view of equation (31) above equation gives

\[ ^*H_a = \tau \sqrt{2 - \tau e^{-\alpha}} H_a \]  

(32)

Thus we have

**Theorem 4.1** A hypersurface \( F^{(n-1)} \) of a Finsler space \( F^n (n > 3) \) is totally geodesic iff the hypersurface \( ^*F^{(n-1)} \) of the space \( ^*F^n \) obtained from \( F^n \) by a projective Matsumoto conformal change, is also totally geodesic. 
5. Hypersurfaces of projectively flat Finsler spaces

In this section, we shall consider a projective Matsumoto conformal change with the Berwald connection $B_l^m$ of $F^n = (M^n, L)$ and $B_l^m$ on $\ast F^n = (\ast M^n, L)$. In the theory of projective changes in the Finsler spaces we have two essential projective invariants, one is the Weyl torsion $W_{ij}^h$ and other is the Douglas tensor $D_{ijk}^h$, so the under projective Matsumoto conformal change, we get $\ast W_{ij}^h = W_{ij}^h$ and $\ast D_{ijk}^h = D_{ijk}^h$.

Now we are concerned with a projectively flat Finsler spaces defined as follows if there exist a projective change $L \rightarrow \ast L$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $\ast F^n = (\ast M^n, L)$ is a locally Minkowski space then $F^n$ is called projectively flat Finsler space. Following two theorems [15] are well-known:

**Theorem 5.1A** Finsler space $F^n (n > 2)$ is projectively flat iff $W_{ij}^h = 0$ and $D_{ijk}^h = 0$.

**Theorem 5.2A** Finsler space $F^n (n > 3)$ is projectively flat then the totally geodesic hypersurfaces $F^{n-1}$ is also projectively flat.

Thus from theorem (18), theorem (5.1) and (5.2), we have

**Theorem 5.3** Let $F^n (n > 3)$ be a projectively flat Finsler space. If the hypersurfaces $F^{n-1}$ is totally geodesic, then the hypersurfaces $\ast F^{(n-1)}$ of the space $\ast F^n$ obtained from $F^n$ by a projective Matsumoto conformal change, is also projectively flat.

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**References**


