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FRACTIONAL INTEGRAL INEQUALITIES INVOLVING BESSEL-MAITLAND FUNCTION AS KERNEL

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Abstract: Fractional integral inequalities play a pivotal role in modern mathematical analysis due to their deep connections with fractional calculus and diverse applications in applied sciences. In this work, we develop a new class of generalized fractional integral inequalities for convex functions by utilizing fractional integral operators characterized through the Bessel–Maitland function. Our approach is framed within the extended Chebyshev functional and relies on the concept of synchronous functions to establish refined inequality bounds. By exploiting the intrinsic structural properties of the Bessel–Maitland kernel, we derive several generalized fractional integral inequalities that unify and extend many existing results associated with classical fractional operators. Furthermore, we demonstrate how the parameters of the Bessel–Maitland function govern the tightness and flexibility of the obtained bounds, thereby offering a broader and more adaptable framework for fractional inequality analysis. The derived fractional integral inequalities can be applied to obtain sharper error bounds and stability estimates in models governed by fractional differential equations arising in physics, engineering, and signal processing.

Keyword: Chebyshev functional, Bessel–Maitland function, convex function and fractional integral inequalities.

1. Introduction

Integral inequalities [32] constitute fundamental tools in mathematical analysis and play a crucial role in the qualitative study of integral and differential equations. Within the framework of fractional calculus, such inequalities are indispensable for examining essential properties of solutions, including boundedness, stability, continuous dependence, and asymptotic behavior. Owing to their wide applicability across science and engineering, fractional integral inequalities have attracted increasing attention in recent decades.

A considerable body of literature has been devoted to the development of fractional integral inequalities using classical and generalized fractional operators (see, for example, [7, 12, 13, 31, 33, 36, 38, 39, 41, 51, 53]). In particular, Grüss-type inequalities and several related results based on Riemann–Liouville fractional integrals were established in [19, 20], while inequalities involving generalized (k, h)-fractional integral operators were introduced in [2]. Furthermore, various Hermite–Hadamard-type inequalities associated with fractional integral operators have been extensively investigated in [8–10, 15, 48, 50]. By employing collections of positive functions, Dahmani [18] derived several fractional integral inequalities, and additional generalizations can be found in [14, 16, 22, 35, 37, 42, 43, 49, 52, 53]. In 2012, Dahmani [17] derived several new fractional results related to convex functions.

Recent advances have focused on the construction of new fractional integral inequalities via newly introduced operators, leading to refined Hermite–Hadamard, Ostrowski, and Jensen-type inequalities under diverse convexity assumptions [1, 21, 23, 25, 29, 45]. Notably, Almoneef et al. [6], Rahman et al. [44], Vivas-Cortez et al. [56], and Talha et al. [55] developed generalized integral inequalities using various fractional operators, including Riemann–Liouville, Atangana–Baleanu, and operators involving special-function kernels. These contributions highlight the growing interplay between fractional operators, convexity theory, and inequality analysis.

Parallel to these developments, significant progress has been achieved in the theory of fractional operators themselves. In particular, Halouani and Bouzeffour [25] introduced and analyzed the fractional Laplace–Bessel operator, providing a rigorous analytical framework that further enriches the operator-theoretic foundations of fractional calculus.

More recently, Sahoo et al. [46, 47], Aljaaidi et al. [5], and Yang et al. [57] obtained notable results involving generalized fractional integral inequalities, while substantial contributions to the theory of integral inequalities were also made by Ngo et al. [36], Pogány [40], Bougoufa [11] and Liu et al. [29]. In [28], Liu et al. established two fundamental theorems concerning integral inequalities, which form an important basis for further investigations.

In recent years, the Bessel–Maitland function has received considerable attention due to its applications in fractional calculus, particularly in defining generalized fractional integral operators and in the study of integral inequalities (see [3, 4, 24, 54, 30]).

Motivated by the recent advances in fractional integral inequalities and the growing interest in generalized fractional operators, as well as by the analytical framework developed for the fractional Laplace–Bessel operator, The primary objective of this work is to develop new fractional integral inequalities associated with generalized Bessel-type fractional operators, extending existing Hermite–Hadamard [26] and Ostrowski-type inequalities [27] reported in recent studies.

Theorem 1: Let $\hbar_1$and $\hbar_2$ be positive continuous functions on $[\phi_1, \phi_2]$, where $\phi_1 < \phi_2$ such that $\hbar_1 \leq \hbar_2$ on $[\phi_1, \phi_2]$. Assume that $\frac{\hbar_1}{\hbar_2}$, $(\hbar_2 \neq 0)$, is decreasing and $\hbar_1$ is an increasing function on $[\phi_1, \phi_2]$. Let $\mathfrak{M}$ be a convex function with $\mathfrak{M}(0) = 0$. Then the following integral inequality holds:

$$\frac{\int_{\phi_1}^{\phi_2} \mathscr{k}_1(z)dz}{\int_{\phi_1}^{\phi_2} \mathscr{k}_2(z)dz} \geq \frac{\int_{\phi_1}^{\phi_2} \mathfrak{M}(\mathscr{k}_1(z))dz}{\int_{\phi_1}^{\phi_2} \mathfrak{M}(\mathscr{k}_2(z))dz} \tag{1}$$

Theorem 2: Let $\mathscr{k}_1$,$\mathscr{k}_2$ and $\mathscr{k}_3$ be positive continuous functions on $[\phi_1, \phi_2]$, where $\phi_1 < \phi_2$ such that $\mathscr{k}_1 \leq \mathscr{k}_2$ on $[\phi_1, \phi_2]$. Assume that $\frac{\mathscr{k}_1}{\mathscr{k}_2}$, ($\mathscr{k}_2 \neq 0$), is decreasing and $\mathscr{k}_1$ and $\mathscr{k}_3$ is an increasing function on $[\phi_1, \phi_2]$. Let $\mathfrak{M}$ be a convex function with $\mathfrak{M}(0) = 0$. Then the following integral inequality holds:

$$\frac{\int_{\phi_1}^{\phi_2} \mathscr{k}_1(z)dz}{\int_{\phi_1}^{\phi_2} \mathscr{k}_2(z)dz} \geq \frac{\int_{\phi_1}^{\phi_2} \mathfrak{M}(\mathscr{k}_1(z))\mathscr{k}_3(z)dz}{\int_{\phi_1}^{\phi_2} \mathfrak{M}(\mathscr{k}_2(z))\mathscr{k}_3(z)dz} \tag{2}$$

Mubeen *et al.* [34] gave the definition of following extended version of generalized Bessel–Maitland function

Definition 1: The extended version of Bessel-Maitland function (EvBMF) is defined for $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1 \in \mathbb{C}$ and $\Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, q, \zeta_1, m, p \geq 0$, as follows:

$$J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(s) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-s)^n}{\Gamma(\alpha_1 n+\varphi_1)(\rho_1)_{mn}(\delta_1)_{pn}} \tag{3}$$

If we replace $s = 1$ in equation (1), then we have

$$J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(1) = \sum_{n=0}^{\infty} \frac{(\eta_1)_{\gamma n}(\xi_1)_{qn}(\nu_1)_{\zeta_1 n}(-1)^n}{\Gamma(\alpha_1 n+\varphi_1)(\rho_1)_{mn}(\delta_1)_{pn}} \tag{4}$$

In [34], Mubeen et al. defined an extension of the fractional integral operator in which the kernel is expressed in terms of an extended version of Bessel–Maitland function.

Definition 2: The fractional integral operator with EvBMF as its kernel is defined as follows:

$$\left(\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;a^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\mathcal{F}\right)(s) = \int_a^s (s-t)^{\varphi_1} J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\omega(s-t)^{\alpha_1})\mathcal{F}(t)dt \tag{5}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, q, \zeta_1, m, p \geq 0$.

Remark 1. By applying (3) $q = 0$, we have the following well known result of [34]

$$\left(\mathfrak{I}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;a^+}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}\mathcal{F}\right)(s) = \int_a^s (s-t)^{\varphi_1} J_{\varphi_1,\gamma,\zeta_1,m,p}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}(\omega(s-t)^{\alpha_1})\mathcal{F}(t)dt \tag{6}$$

Remark 2. If we assume $\omega = 0$ and replace φ_1 by $\varphi_1 - 1$ then we get the left-sided Riemann-Liouville fractional operator.

$$\left(\mathfrak{I}_{a^+}^{\varphi_1}\mathcal{F}\right)(s) = \frac{1}{\Gamma(\varphi_1)} \int_a^s (s-t)^{\varphi_1 - 1} \mathcal{F}(t)dt \tag{7}$$

In this work, we derives fractional integral inequalities for the extended Chebyshev functional in the framework of synchronous functions using an appropriate integral operator (5).

2. Fractional Integral Inequalities for Convex Functions Associated with the Bessel–Maitland Function

This section derives fractional integral inequalities for convex functions using a fractional integral operator whose kernel involves the extended Bessel–Maitland function.

Theorem 3: Let $\hbar_1$and $\hbar_2$ are two positive and continuous functions on $[\phi_1, \phi_2]$ and $\hbar_1 \leq \hbar_2$ on $[\phi_1, \phi_2]$. Suppose that $\frac{\hbar_1}{\hbar_2}$, $(\hbar_2 \neq 0)$, is decreasing and $\hbar_1$ is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\hbar_1(s))}{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\hbar_2(s))} \geq \frac{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(\hbar_1(s))\right)}{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(\hbar_2(s))\right)} \tag{8}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}\, with\ s \geq 0, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1,\ \eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1),\ \Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, q, \zeta_1, m, p \geq 0$.

Proof: Since the function $\mathfrak{M}$ is convex such that $\mathfrak{M}(0) = 0$ then the function $\frac{\mathfrak{M}(s)}{s}$ is increasing. Also $\hbar_1$ is increasing therefore, $\frac{\mathfrak{M}(\hbar_1(s))}{\hbar_1(s)}$ is also increasing. Again $\frac{\hbar_1(s)}{\hbar_2(s)}$ is decreasing. Then for all $\mu, \phi \in [\phi_1, \phi_2]$, we have

$$\left(\frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)} - \frac{\mathfrak{M}(\hbar_1(\phi))}{\hbar_1(\phi)}\right)\left(\frac{\hbar_1(\phi)}{\hbar_2(\phi)} - \frac{\hbar_1(\mu)}{\hbar_2(\mu)}\right) \geq 0 \tag{9}$$

It can be written as

$$\frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)}\frac{\hbar_1(\phi)}{\hbar_2(\phi)} + \frac{\mathfrak{M}(\hbar_1(\phi))}{\hbar_1(\phi)}\frac{\hbar_1(\mu)}{\hbar_2(\mu)} - \frac{\mathfrak{M}(\hbar_1(\phi))}{\hbar_1(\phi)}\frac{\hbar_1(\phi)}{\hbar_2(\phi)} - \frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)}\frac{\hbar_1(\mu)}{\hbar_2(\mu)} \geq 0 \tag{10}$$

Now multiply (10) by $\hbar_2(\mu)\, \hbar_2(\phi)$ then

$$\frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)}\hbar_1(\phi)\hbar_2(\mu) + \frac{\mathfrak{M}(\hbar_1(\phi))}{\hbar_1(\phi)}\hbar_1(\mu)\hbar_2(\phi) - \frac{\mathfrak{M}(\hbar_1(\phi))}{\hbar_1(\phi)}\hbar_1(\phi)\hbar_2(\mu)$$

$$-\frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)}\hbar_1(\mu)\hbar_2(\phi) \geq 0 \tag{11}$$

Now multiplying (11) by $(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\,\omega(s-\mu)^{\alpha_1})$ and then integrating (11) with respect to μ over $[\phi_1, s], \phi_1 < s < \phi_2$, we get

$$\int_{\phi_1}^{s} (s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\,\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\hbar_1(\mu))}{\hbar_1(\mu)}\hbar_1(\phi)\hbar_2(\mu)d\mu$$

$$+\int_{\phi_1}^{s}(s-\mu)^{\varphi_1}J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\,\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathscr{k}_1(\phi))}{\mathscr{k}_1(\phi)}\mathscr{k}_1(\mu)\mathscr{k}_2(\phi)d\mu$$

$$-\int_{\phi_1}^{s}(s-\mu)^{\varphi_1}J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\,\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathscr{k}_1(\phi))}{\mathscr{k}_1(\phi)}\mathscr{k}_1(\phi)\mathscr{k}_2(\mu)\,d\mu$$

$$-\int_{\phi_1}^{s}(s-\mu)^{\varphi_1}J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\,\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathscr{k}_1(\mu))}{\mathscr{k}_1(\mu)}\mathscr{k}_1(\mu)\mathscr{k}_2(\phi)d\mu\geq 0 \tag{12}$$

Using (5) we have

$$\mathscr{k}_1(\phi)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_2(s)\right)$$

$$+\frac{\mathfrak{M}(\mathscr{k}_1(\phi))}{\mathscr{k}_1(\phi)}\mathscr{k}_2(\phi)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_1(s)\big)$$

$$-\frac{\mathfrak{M}(\mathscr{k}_1(\phi))}{\mathscr{k}_1(\phi)}\mathscr{k}_1(\phi)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_2(s)\big)$$

$$-\mathscr{k}_2(\phi)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_1(s)\right)\geq 0 \tag{13}$$

Now multiplying (13) by $(s-\phi)^{\varphi_1}J_{\varphi_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\,\omega(s-\phi)^{\alpha_1})$ and then integrating (13) with respect to ϕ over $[\phi_1,s],\phi_1<s<\phi_2$, we get

$$\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_1(s)\big)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_2(s)\right)$$

$$+\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_2(s)\right)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_1(s)\big)$$

$$-\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_1(s)\right)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_2(s)\big)$$

$$-\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_2(s)\big)\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_1(s)\right)\geq 0$$

$$\Rightarrow\frac{\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_1(s)\big)}{\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\big(\mathscr{k}_2(s)\big)}\geq\frac{\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_1(s)\right)}{\mathfrak{J}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathscr{k}_1(s))}{\mathscr{k}_1(s)}\mathscr{k}_2(s)\right)} \tag{14}$$

Since$\mathscr{k}_1\leq\ \mathscr{k}_2$ on $[\phi_1,\phi_2]$ and $\frac{\mathfrak{M}(s)}{s}$ is an increasing function, for $\mu\in[\phi_1,s],\phi_1<s<\phi_2$, we have

$$\frac{\mathfrak{M}(\mathscr{k}_1(\mu))}{\mathscr{k}_1(\mu)}\ \leq\ \frac{\mathfrak{M}(\mathscr{k}_2(\mu))}{\mathscr{k}_2(\mu)} \tag{15}$$

Now multiplying (15) by $(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-\mu)^{\alpha_1})$ and then integrating (15) with respect to μ over $[\phi_1, s], \phi_1 < s < \phi_2$, we get

$$\int_{\phi_1}^{s} (s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1}) \frac{\mathfrak{M}(k_1(\mu))}{k_1(\mu)} k_2(\mu) d\mu$$

$$\leq \int_{\phi_1}^{s} (s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1}) \frac{\mathfrak{M}(k_2(\mu))}{k_2(\mu)} k_2(\mu) d\mu \tag{16}$$

Using equation (5) we get

$$\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(k_1(s))}{k_1(s)} k_2(s)\right) \leq \mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right) \tag{17}$$

By combining (17) and (14) we get the desired result.

By setting $q = 0$ in Theorem 3, we obtain the following result for the operator defined in (6).

Corollary 4: Let $k_1$and k_2 are two positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}(k_2(s))} \geq \frac{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))\right)}{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right)} \tag{18}$$

where $\eta_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}\, with\ s \geq 0, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \eta_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1),\ \Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, \zeta_1, m, p \geq 0$.

Remark 3. Setting $\omega = 0$ in Theorem 3, yields the result due to Dahmani [17].

Remark 4. By applying Theorem 3 with $\omega = 0, \varphi_1 = 1$ and $s = \phi_2$, we obtain Theorem 1.

Theorem 5: Let $k_1$and k_2 are two positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$ $(\phi_1 < \phi_2)$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right)+\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))\right)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,,\omega;\phi_1^+}(k_2(s))+\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(k_2(s))\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))\right)} \geq 1 \tag{19}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \tau_1, \omega \in \mathbb{C}\, with\ S \geq 0, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \Re(\tau_1) \geq -1, \eta_1\ , \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1),\ \Re(\eta_1) > 0, \Re(\xi_1) > 0,\ \Re(\nu_1) > 0,\ \Re(\rho_1) > 0,\ \Re(\delta_1) > 0$ and $\gamma, q, \zeta_1, m, p \geq 0$.

Proof: Since the function $\mathfrak{M}$ is convex such that $\mathfrak{M}(0)=0$ then the function $\frac{\mathfrak{M}(s)}{s}$ is increasing. Also $\mathcal{k}_1$ is increasing therefore, $\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}$ is also increasing. Again $\frac{\mathcal{k}_1(s)}{\mathcal{k}_2(s)}$ is decreasing. Then for all $\phi \in [\phi_1, s]$, $\phi_1 < s \le \phi_2$. Thus multiplying (13) by $(s-\phi)^{\tau_1} J_{\tau_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\omega(s-\phi)^{\alpha_1})$ and then integrating (9) with respect to ϕ over $[\phi_1, s], \phi_1 < s \le \phi_2$, we get

$$\mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\mathcal{k}_1(s))\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\right)$$

$$+\mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\right)\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\mathcal{k}_1(s))$$

$$\ge \mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_1(s)\right)\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\mathcal{k}_2(s))$$

$$+\mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\mathcal{k}_2(s))\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_1(s)\right) \quad (20)$$

Since the function $\mathcal{k}_1 \le \mathcal{k}_2$ on $[\phi_1, \phi_2]$ and $\frac{\mathfrak{M}(s)}{s}$ is an increasing function, for $\phi \in [\phi_1, s], \phi_1 < s < \phi_2$, we have

$$\frac{\mathfrak{M}(\mathcal{k}_1(\phi))}{\mathcal{k}_1(\phi)} \le \frac{\mathfrak{M}(\mathcal{k}_2(\phi))}{\mathcal{k}_2(\phi)} \quad (21)$$

Multiplying (21) by $(s-\phi)^{\tau_1} J_{\tau_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\omega(s-\phi)^{\alpha_1})$ and then integrating (21) with respect to ϕ over $[\phi_1, s], \phi_1 < s \le \phi_2$, we get

$$\int_{\phi_1}^{s}(s-\phi)^{\tau_1} J_{\tau_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\omega(s-\phi)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_1(\phi))}{\mathcal{k}_1(\phi)}\mathcal{k}_2(\phi)d\phi$$

$$\le \int_{\phi_1}^{s}(s-\phi)^{\tau_1} J_{\tau_1,\gamma,q,\zeta_1,m,p}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}(\omega(s-\phi)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_2(\phi))}{\mathcal{k}_2(\phi)}\mathcal{k}_2(\phi)d\phi \quad (22)$$

Using (5) we get

$$\mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\right) \le \mathfrak{I}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\mathfrak{M}(\mathcal{k}_2(s))\right) \quad (23)$$

Similarly, we can get

$$\mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\right) \le \mathfrak{I}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}\left(\mathfrak{M}(\mathcal{k}_2(s))\right) \quad (24)$$

By combining (20), (23) and (24) we get the desired result.

By setting $q=0$ in Theorem 5, we obtain the following result for the operator defined in (6).

Corollary 6: Let $k_1$and k_2 are two positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$ $(\phi_1 < \phi_2)$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_2(s))\big)+\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_2(s))\big)\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_1(s))\big)\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,,\omega;\phi_1^+}(k_2(s))+\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}(k_2(s))\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_1(s))\big)} \geq 1 \tag{25}$$

where $\eta_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \tau_1, \omega \in \mathbb{C}\, with\ S \geq 0, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \Re(\tau_1) \geq -1, \eta_1$ $, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0$, $\Re(\nu_1) > 0$, $\Re(\rho_1) > 0$, $\Re(\delta_1) > 0$ and $\gamma, \zeta_1, m, p \geq 0$.

Remark 5. Setting $\omega = 0$ in Theorem 5, yields the result due to Dahmani [17].

Remark 6. By applying Theorem 5 with $\omega = 0, \tau_1 = \varphi_1 = 1$ and $s = \phi_2$, we obtain Theorem 1.

Remark 7. By applying Theorem 5 with $\tau_1 = \varphi_1$, we obtain Theorem 3.

Theorem 7: Let k_1,k_2 and k_3 are positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$ $(\phi_1 < \phi_2)$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 and k_3 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(k_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(k_2(s))} \geq \frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_1(s))k_3(s)\big)}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\big(\mathfrak{M}(k_2(s))k_3(s)\big)} \tag{26}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, q, \zeta_1, m, p \geq 0$.

Proof: Since the function $\mathfrak{M}$ is convex such that $\mathfrak{M}(0) = 0$ then the function $\frac{\mathfrak{M}(s)}{s}$ is increasing. Also k_1 is increasing therefore, $\frac{\mathfrak{M}(k_1(s))}{k_1(s)}$ is also increasing. Again $\frac{k_1(s)}{k_2(s)}$ is decreasing. Then for all $\mu, \phi \in [\phi_1, s]$, $\phi_1 < s \leq \phi_2$, we have

$$\left(\frac{\mathfrak{M}(k_1(\mu))}{k_1(\mu)}k_3(\mu) - \frac{\mathfrak{M}(k_1(\phi))}{k_1(\phi)}k_3(\phi)\right).\big(k_1(\phi)k_2(\mu) - k_1(\mu)k_2(\phi)\big) \geq 0 \tag{27}$$

$$\Rightarrow \frac{\mathfrak{M}(k_1(\mu))k_3(\mu)}{k_1(\mu)}k_1(\phi)k_2(\mu) + \frac{\mathfrak{M}(k_1(\phi))k_3(\phi)}{k_1(\phi)}k_1(\mu)k_2(\phi)$$

$$-\frac{\mathfrak{M}(k_1(\phi))k_3(\phi)}{k_1(\phi)}k_1(\phi)k_2(\mu) - \frac{\mathfrak{M}(k_1(\mu))k_3(\mu)}{k_1(\mu)}k_1(\mu)k_2(\phi) \geq 0 \tag{28}$$

Now multiplying (28) by $(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-\mu)^{\alpha_1})$ and then integrating (28) with respect to μ over $[\phi_1, s], \phi_1 < s \leq \phi_2$, we get

$$\int_{\phi_1}^{s}(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_1(\mu))\mathcal{k}_3(\mu)}{\mathcal{k}_1(\mu)}\mathcal{k}_1(\phi)\mathcal{k}_2(\mu)d\mu$$

$$+\int_{\phi_1}^{s}(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_1(\phi))\mathcal{k}_3(\phi)}{\mathcal{k}_1(\phi)}\mathcal{k}_1(\mu)\mathcal{k}_2(\phi)d\mu$$

$$-\int_{\phi_1}^{s}(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_1(\phi))\mathcal{k}_3(\phi)}{\mathcal{k}_1(\phi)}\mathcal{k}_1(\phi)\mathcal{k}_2(\mu)\,d\mu$$

$$-\int_{\phi_1}^{s}(s-\mu)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-t)^{\alpha_1})\frac{\mathfrak{M}(\mathcal{k}_1(\mu))\mathcal{k}_3(\mu)}{\mathcal{k}_1(\mu)}\mathcal{k}_1(\mu)\mathcal{k}_2(\phi)d\mu \geq 0 \quad (29)$$

Using (5) we get

$$\mathcal{k}_1(\phi)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\mathcal{k}_3(s)\right)$$

$$+\frac{\mathfrak{M}(\mathcal{k}_1(\phi))}{\mathcal{k}_1(\phi)}\mathcal{k}_2(\phi)\mathcal{k}_3(\phi)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_1(s))$$

$$-\frac{\mathfrak{M}(\mathcal{k}_1(\phi))}{\mathcal{k}_1(\phi)}\mathcal{k}_1(\phi)\mathcal{k}_3(\phi)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_2(s))$$

$$-\mathcal{k}_2(\phi)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_1(s)\mathcal{k}_3(s)\right) \geq 0 \quad (30)$$

Now multiplying (30) by $(s-\phi)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-\phi)^{\alpha_1})$ and then integrating (30) with respect to ϕ over $[\phi_1, s], \phi_1 < s < \phi_2$, we get

$$\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_1(s))\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\mathcal{k}_3(s)\right)$$

$$+\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\mathcal{k}_3(s)\right)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_1(s))$$

$$\geq \mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_1(s)\mathcal{k}_3(s)\right)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_2(s))$$

$$+\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_2(s))\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_1(s)\mathcal{k}_3(s)\right) \quad (31)$$

It can be written as

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}(\mathcal{k}_2(s))} \geq \frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(\mathcal{k}_1(s))\mathcal{k}_3(s)\right)}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(\mathcal{k}_1(s))}{\mathcal{k}_1(s)}\mathcal{k}_2(s)\mathcal{k}_3(s)\right)} \quad (32)$$

Now using (17) and (32) we get the desired result.

By setting $q = 0$ in Theorem 7, we obtain the following result for the operator defined in (6).

Corollary 8: Let k_1,k_2 and k_3 are positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$ $(\phi_1 < \phi_2)$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 and k_3 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{J}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(k_1(s)\right)}{\mathfrak{J}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(k_2(s)\right)} \geq \frac{\mathfrak{J}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))k_3(s)\right)}{\mathfrak{J}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))k_3(s)\right)} \tag{33}$$

where $\eta_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1$, $\eta_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$ and $\gamma, \zeta_1, m, p \geq 0$.

Remark 8. Setting $\omega = 0$ in Theorem 7, yields the result due to Dahmani [17].

Remark 9. By applying Theorem 7 with $\omega = 0, \varphi_1 = 1$ and $s = \phi_2$, we obtain Theorem 2.

Theorem 9: Let k_1,k_2 and k_3 are positive and continuous functions on $[\phi_1, \phi_2]$, where$(\phi_1 < \phi_2)$ and assume that$k_1 \leq k_2$ on $[\phi_1, \phi_2]$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is a decreasing function and k_1 and k_3 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_1(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right)+\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_2(s))\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_1(s)\right)}{\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))k_3(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_2(s)\right)+\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_2(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathfrak{M}(k_1(s))k_3(s)\right)} \geq 1 \tag{34}$$

where $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \tau_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \Re(\tau_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0$, $\gamma, q, \zeta_1, m, p \geq 0$.

Proof: Multiplying (30) by $(s - \phi)^{\tau_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p}(\,\omega(s - \phi)^{\alpha_1})$ and then integrating (30) with respect to ϕ over $[\phi_1, s], \phi_1 < s \leq \phi_2$, we get

$$\begin{aligned}
&\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_1(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(k_1(s))}{k_1(s)}k_2(s)k_3(s)\right)\\
&+\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(k_1(s))}{k_1(s)}k_2(s)k_3(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_1(s)\right)\\
&\geq \mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(k_1(s))}{k_1(s)}k_1(s)k_3(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_2(s)\right)\\
&+\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(k_2(s)\right)\mathfrak{J}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\frac{\mathfrak{M}(k_1(s))}{k_1(s)}k_1(s)k_3(s)\right)
\end{aligned} \tag{35}$$

Using (23), (24) and (35) we get the desired result.

By setting $q = 0$ in Theorem 9, we obtain the following result for the operator defined in (6).

Corollary 10: Let k_1,k_2 and k_3 are positive and continuous functions on $[\phi_1, \phi_2]$ and $k_1 \leq k_2$ on $[\phi_1, \phi_2]$ $(\phi_1 < \phi_2)$. Suppose that $\frac{k_1}{k_2}$, $(k_2 \neq 0)$, is decreasing and k_1 and k_3 is an increasing function on $[\phi_1, \phi_2]$. Then for any convex function $\mathfrak{M}$ with $\mathfrak{M}(0) = 0$, the following integral inequality holds:

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}(k_1(s))\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}\big(\mathfrak{M}(k_2(s))\big)+\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}\big(\mathfrak{M}(k_2(s))\big)\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}(k_1(s))}{\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}\big(\mathfrak{M}(k_1(s))k_3(s)\big)\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}(k_2(s))+\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}(k_2(s))\mathfrak{I}^{\alpha_1,\eta_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,\zeta_1,m,p,\omega;\phi_1{}^+}\big(\mathfrak{M}(k_1(s))k_3(s)\big)} \geq 1 \quad (36)$$

where $\eta_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \tau_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \Re(\tau_1) \geq -1$, $\eta_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1)$, $\Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0, \gamma, \zeta_1, m, p \geq 0$.

Remark 10. Setting $\omega = 0$ in Theorem 9, yields the result due to Dahmani [17].

Remark 11. By applying Theorem 9 with $\omega = 0, \tau_1 = \varphi_1 = 1$ and $s = \phi_2$, we obtain Theorem 2.

Remark 12. By applying Theorem 9 with $\tau_1 = \varphi_1$, we obtain Theorem 7.

3. Further Results on Fractional Integral Inequalities Associated with the Bessel–Maitland Function

This section presents additional generalized fractional integral inequalities whose kernels are defined in terms of the Bessel–Maitland function.

Theorem 11: Let $k_1$and k_2 are two positive and continuous functions on $[\phi_1, \phi_2]$ such that k_1 is decreasing and k_2 is an increasing function on $[\phi_1, \phi_2]$. Then for $s \geq 0$ $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1$, $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1), \Re(\eta_1)$
$0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0, \gamma, q, \zeta_1, m, p \geq 0$, $u \geq v > 0$ and $w > 0$ we have

$$\frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_1}^u(s)\right)}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_1}^v(s)\right)} \geq \frac{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_2}^w {k_1}^u(s)\right)}{\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_2}^w {k_1}^v(s)\right)} \quad (37)$$

Proof: Assume that $\mu, \phi \in [\phi_1, \phi_2]$, we have

$$\left({k_2}^w(\mu) - {k_2}^w(\phi)\right)\left({k_1}^u(\phi){k_1}^v(\mu) - {k_1}^v(\phi){k_1}^u(\mu)\right) \geq 0$$

which implies that

$${k_2}^w(\mu){k_1}^u(\phi){k_1}^v(\mu) + {k_2}^w(\phi){k_1}^v(\phi){k_1}^u(\mu)$$
$$\geq {k_2}^w(\mu){k_1}^v(\phi){k_1}^u(\mu) + {k_2}^w(\phi){k_1}^u(\phi){k_1}^v(\mu) \quad (38)$$

Now multiplying (38) both side by $(s-\phi)^{\varphi_1} J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\omega(s-\phi)^{\alpha_1})$ and then integrating (38) with respect to ϕ over $[\phi_1, s]$, $\phi_1 < s < \phi_2$, we get

$${k_2}^w(\mu){k_1}^v(\mu)\, \mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_1}^u(s)\right) + {k_1}^u(\mu)\mathfrak{I}^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1{}^+}\left({k_2}^w {k_1}^v(s)\right)$$

$$\geq \mathscr{k}_2^{\,w}(\mu)\mathscr{k}_1^{\,u}(\mu)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right) + \mathscr{k}_1^{\,v}(\mu)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right) \qquad (39)$$

Again multiplying (39) both side by $(s-\mu)^{\varphi_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\,\omega(s-\mu)^{\alpha_1})$ and then integrating (39) with respect to μ over $[\phi_1,s], \phi_1 < s < \phi_2$, we get

$$\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)\,\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)$$

$$+\,\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)$$

$$\geq \Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)$$

$$+\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)$$

which gives the result.

Theorem 12: Let $\mathscr{k}_1$and $\mathscr{k}_2$ are two positive and continuous functions on $[\phi_1,\phi_2]$ such that $\mathscr{k}_1$ is decreasing and $\mathscr{k}_2$ is an increasing function on $[\phi_1,\phi_2]$. Then for $s \geq 0\,, \eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1,$

$\tau_1, \omega \in \mathbb{C}\,, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1, \Re(\tau_1) \geq -1,\ \eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1),\ \ \Re(\,\eta_1) > 0,$

$\Re(\xi_1) > 0,\ \Re(\nu_1) > 0,\ \Re(\rho_1) > 0,\ \Re(\delta_1) > 0, \gamma, q, \zeta_1, m, p \geq 0, u \geq v > 0$ and $w > 0$ we have

$$\frac{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)+\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)}{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)+\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)} \geq 1 \quad (40)$$

Proof: Multiplying (38) by $(s-\mu)^{\tau_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p}(\,\omega(s-\mu)^{\alpha_1})$ and then integrating (38) with respect to μ over $[\phi_1,s], \phi_1 < s \leq \phi_2$, we get

$$\mathscr{k}_1^{\,u}(\phi)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right) + \mathscr{k}_2^{\,w}(\phi)\mathscr{k}_1^{\,v}(\phi)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)$$

$$\geq \mathscr{k}_1^{\,v}(\phi)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right) + \mathscr{k}_2^{\,w}(\phi)\mathscr{k}_1^{\,u}(\phi)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right) \qquad (41)$$

Again multiplying (41) by$(s-\phi)^{\varphi_1}J^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p}(\,\omega(s-\phi)^{\alpha_1})$ and then integrating (41) with respect to ϕ over $[\phi_1,s], \phi_1 < s \leq \phi_2$, we get

$$\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)$$

$$+\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,v}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,u}(s)\right)$$

$$\geq \Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)$$

$$+\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2^{\,w}\mathscr{k}_1^{\,u}(s)\right)\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\tau_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1^{\,v}(s)\right)$$

which gives the result.

Theorem 13: Let $\mathscr{k}_1$and $\mathscr{k}_2$ are two positive and continuous functions on $[\phi_1, \phi_2]$ where $\mathscr{k}_1$ is decreasing and $\mathscr{k}_2$ is an increasing function on $[\phi_1, \phi_2]$, such that

$$\left(\mathscr{k}_1{}^{w}(\phi)\mathscr{k}_2{}^{w}(\mu) - \mathscr{k}_1{}^{w}(\mu)\mathscr{k}_2{}^{w}(\phi)\right)\left(\mathscr{k}_1{}^{w}(\phi) - \mathscr{k}_2{}^{w}(\mu)\right) \geq 0 \quad (42)$$

Then for $s \geq 0$ $\eta_1, \xi_1, \nu_1, \alpha_1, \varphi_1, \rho_1, \delta_1, \omega \in \mathbb{C}, \Re(\alpha_1) \geq 0, \Re(\varphi_1) \geq -1,$ $\eta_1, \xi_1, \nu_1 < \rho_1 + \delta_1 + \Re(\alpha_1), \Re(\eta_1) > 0, \Re(\xi_1) > 0, \Re(\nu_1) > 0, \Re(\rho_1) > 0, \Re(\delta_1) > 0, \gamma, q, \zeta_1, m, p \geq 0, u \geq v$

> 0 and $w > 0$ we have

$$\frac{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1{}^{w+u}(s)\right)}{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_1{}^{w+v}(s)\right)} \geq \frac{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2{}^{w}\mathscr{k}_1{}^{u}(s)\right)}{\Im^{\alpha_1,\eta_1,\xi_1,\nu_1,\rho_1,\delta_1}_{\varphi_1,\gamma,q,\zeta_1,m,p,\omega;\phi_1^+}\left(\mathscr{k}_2{}^{w}\mathscr{k}_1{}^{v}(s)\right)} \quad (43)$$

Proof: Assume that $\mu, \phi \in [\phi_1, \phi_2]$, we have

$$\left(\mathscr{k}_1{}^{w}(\phi)\mathscr{k}_2{}^{w}(\mu) - \mathscr{k}_1{}^{w}(\mu)\mathscr{k}_2{}^{w}(\phi)\right)\left(\mathscr{k}_1{}^{w}(\phi) - \mathscr{k}_2{}^{w}(\mu)\right) \geq 0$$

and using the same process as in theorem 11, we get the desired result.

4. Conclusion

In this work, new generalized fractional integral inequalities have been established using an integral operator whose kernel involves the extended Bessel–Maitland function. The results provide a unified framework for deriving several well-known inequalities under suitable convexity and monotonicity conditions. By appropriate choices of parameters, many existing results in the literature are recovered as special cases, demonstrating the generality of the proposed approach. Moreover, additional inequalities involving the extended Bessel–Maitland function are obtained, which further enrich the theory of fractional integral inequalities. These findings contribute to the ongoing development of fractional calculus and may be useful in future analytical and applied studies.

5. Application

The derived fractional integral inequalities can be applied to obtain bounds, stability criteria, and qualitative estimates for solutions of fractional differential and integral equations involving nonlocal kernels.

6. Future Scope

Future work may extend these results to other classes of convexity, multi-variable fractional operators, or different special-function kernels, with potential applications in mathematical physics and applied sciences.

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