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q-ANALOGUE PROPERTIES OF MEROMORPHIC MULTIVALENT FUNCTIONS VIA THREE-CUSPED EPICYCLOID SUBORDINATION

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Abstract : In this article, we introduce two Ma–Minda type subclasses, denoted by $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and $\mathcal{MK}_{p,q}[b;\mathcal{L}]$, of q-starlike and q-convex meromorphic multivalent functions, respectively, defined in the punctured unit disk. These subclasses are established using the concept of subordination between analytic functions. The classes $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and $\mathcal{MK}_{p,q}[b;\mathcal{L}]$ are constructed via the q-derivative operator and an analytic function $\mathcal{L}(z) = 1 + \frac{4z}{5} + \frac{1}{5}z^4$, which maps the unit disk onto a three-petal leaf-shaped bounded region that is symmetric about the real axis and contained in the right half of complex plane. Characterizations of these subclasses are derived using Hadamard (convolution) products. Moreover, we investigate convolution properties, establish necessary and sufficient conditions, and obtain coefficient bounds for functions belonging to the proposed classes.

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1. Introduction and Preliminaries

Let Σ_p denote the class of meromorphic multivalent functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p}, \quad (p \in \mathbb{N}) \tag{1}$$

which are analytic in the punctured unit disc $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. For $p = 1$, we denote $\Sigma_1 = \Sigma$.

Also an analytic function g is said to be *subordinate* to an analytic function f in $\mathbb{U}$, written as $g \prec f$, if there exists a Schwarz function w such that $g(z) = f(w(z))$ for $z \in \mathbb{U}$. If f is univalent in $\mathbb{U}$, this subordination is equivalent to $g(0) = f(0)$ and $g(\mathbb{U}) \subseteq f(\mathbb{U})$.

In recent years, q-calculus has emerged as a powerful tool in various scientific disciplines, including physics and statistics. In Geometric Function Theory, Jackson's q-derivative operator ([1], [2], [3]) has been extensively utilized to define novel subclasses of analytic and meromorphic functions (see [4],[5],[6], [7], [8], [9], [10], [11], [12] [13], [14]and references therein). For $0 < q < 1$, the q-derivative of a function f is defined by:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0), \tag{2}$$

with $D_q f(0) = f'(0)$. For $f \in \Sigma_p$, the q-derivative is given by:

$$D_q f(z) = \frac{-[p]_q}{q^p z^{p+1}} + \sum_{k=1}^{\infty} [k-p]_q a_k z^{k-p-1}, \tag{3}$$

where $[n]_q = \frac{1-q^n}{1-q}$ denotes the q-number. As $q \to 1^-$, $D_q f(z) \to f'(z)$. Motivated by the Ma–Minda [15] framework and the recent work of several authors ([16], [4]. [17], [18], [19]),we introduce two subclasses of meromorphic functions associated with the function $\mathcal{L}(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4$. This function maps $\mathbb{U}$ onto a region bounded by a three-cusped epicycloid (a three-petaled leaf-like domain).

Definition 1. *For $b \in \mathbb{C}\backslash\{0\}$, the classes $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and $\mathcal{MK}_{p,q}[b;\mathcal{L}]$ are defined by:*

$$\mathcal{MS}^*_{p,q}[b;\mathcal{L}] = \left\{ f \in \Sigma_p : 1 - \frac{1}{b}\left[\frac{zD_q f(z)}{f(z)} + \frac{[p]_q}{q^p}\right] \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}, \tag{4}$$

and

$$\mathcal{MK}_{p,q}[b;\mathcal{L}] = \left\{ f \in \Sigma_p : 1 - \frac{1}{b}\left[\frac{D_q(zD_q f(z))}{D_q f(z)} + \frac{[p]_q}{q^p}\right] \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}. \tag{5}$$

It follows immediately from the definitions that:

$$f \in \mathcal{MK}_{p,q}[b;\mathcal{L}] \Leftrightarrow -\frac{q^p}{[p]_q} zD_q f \in \mathcal{MS}^*_{p,q}[b;\mathcal{L}]. \tag{6}$$

The following particular cases may be noted.

As $q \to 1$ and $p = 1$,

(i) $\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}[b;\mathcal{L}] = \mathcal{MS}^*[b;\mathcal{L}] = \left\{ f \in \Sigma : 1 - \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + 1\right] \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}$

(ii) $\lim_{q\to 1^-} \mathcal{MK}_{1,q}[b;\mathcal{L}] = \mathcal{MK}[b;\mathcal{L}] = \left\{ f \in \Sigma : 1 - \frac{1}{b}\left[\frac{zf''(z))}{f'(z)} + 1\right] \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}$.

As $q \to 1$, $p = 1$ and $b = 1$,

(iii) $\lim_{q\to 1^-} \mathcal{MS}^*_{1,q}[1;\mathcal{L}] = \mathcal{MS}^*[\mathcal{L}] = \left\{ f \in \Sigma : \frac{-zf'(z)}{f(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}$

(iv) $\lim_{q\to 1^-} \mathcal{MK}_{1,q}[b;\mathcal{L}] = \mathcal{MK}[\mathcal{L}] = \left\{ f \in \Sigma : -1 - \frac{zf''(z))}{f'(z)} \prec 1 + \frac{4}{5}z + \frac{1}{5}z^4 \right\}$.

For $f, g \in \Sigma_p$, the Hadamard product (or convolution) $(f * g)(z)$ is defined by

$z^{-p}+\sum_{k=1}^{\infty} a_k\, b_k z^{k-p}$.

By the definition, it is easy to verify that for any function $f\in\Sigma_p$

$$f(z)*\frac{1}{z^p(1-z)}=f(z) \tag{7}$$

and

$$f(z)*\frac{1-\left(q+\frac{1}{[p]_q}\right)z}{z^p(1-z)(1-qz)}=-\frac{q^p}{[p]_q}zD_qf(z). \tag{8}$$

This operator is central to our subsequent characterizations, necessary and sufficient conditions, and coefficient estimates for the defined subclasses.

2. Hadamard Product Properties

Unless otherwise mentioned, we assume throughout this section that $0<q<1$, and $\theta\in[0,2\pi)$.

Theorem 2.1. If $f\in\Sigma_p$, then $f\in\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ if and only if

$$z^p\left[f(z)*\frac{1+[\Lambda(\theta)-q]z}{z^p(1-z)(1-qz)}\right]\neq 0 \qquad (z\in\mathbb{U}^*), \tag{9}$$

where

$$\Lambda(\theta)=\frac{5}{bq^p(4e^{i\theta}+e^{4i\theta})}. \tag{10}$$

Proof. First, let $f\in\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$, then in order to prove (9) we will write (4) by using the definition of the subordination, that is

$$-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)}=1+\frac{bq^p\left(4w(z)+w^4(z)\right)}{5[p]_q}, \quad (z\in\mathbb{U}^*) \tag{11}$$

where w is a Schwarz function, hence

$$z^p\left[-q^pzD_qf(z)-\{[p]_q+\frac{bq^p}{5}(4e^{i\theta}+e^{4i\theta})\}f(z)\right]\neq 0\ , \qquad (z\in\mathbb{U}^*). \tag{12}$$

Now from (7) and (8), we may write (11) as

$$z^p[f(z)*\frac{\left\{1-\left(q+\frac{1}{[p]_q}\right)z\right\}[p]_q}{z^p(1-z)(1-qz)}$$

$$-\left\{[p]_q+\frac{bq^p}{5}(4e^{i\theta}+e^{4i\theta})\right\}\left(f(z)*\frac{1}{z^p(1-z)}\right)]\neq 0 \qquad (z\in\mathbb{U}^*),$$

which is equivalent to

$$z^p\left[f(z)*\frac{1+\left(-q+\frac{5}{bq^p(4e^{i\theta}+e^{4i\theta})}\right)z}{z^p(1-z)(1-qz)}[-\frac{bq^p}{5}(4e^{i\theta}+e^{4i\theta})]\right]\neq 0$$

or

$$z^p\left[f(z)*\frac{1+\left(\frac{5}{bq^p(4e^{i\theta}+e^{4i\theta})}-q\right)z}{z^p(1-z)(1-qz)}\right]\neq 0, \qquad (z\in\mathbb{U}^*) \tag{13}$$

which is the required condition (9). This proves the necessary part of Theorem 2.1. Conversely, suppose that $f\in\Sigma_p$ satisfy the condition (9). Since it was shown in the first part of the proof that assumption (11) is equivalent to (9), we obtain that

$$-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)}\neq 1+\frac{bq^p(4e^{i\theta}+e^{4i\theta})}{5[p]_q}, \qquad (z\in\mathbb{U}^*) \tag{14}$$

Let us assume that

$$\varphi(z)=-\frac{q^p}{[p]_q}\frac{zD_qf(z)}{f(z)} \quad and \quad \psi(z)=1+\frac{bq^p(4z+z^4)}{5[p]_q}.$$

The relation (14) means that

$$\varphi(\mathbb{U}^*)\cap\psi(\partial\mathbb{U}^*)=\emptyset.$$

Thus, the simply connected domain is included in a connected component of $\mathbb{C}\setminus\psi(\partial\mathbb{U}^*)$. Therefore, using the fact that $\varphi(0)=\psi(0)$ and the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, which implies that $f\in\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$. Thus, the proof of Theorem 2.1 is completed. □

Theorem 2.2. *If $f\in\Sigma_p$, then $f\in\mathcal{MK}_{p,q}[b;\mathcal{L}]$ if and only if*

$$z^p\left[f(z)*\frac{1-\left[\frac{1+\Lambda(\theta)}{[p]_q}-(\Lambda(\theta)-q-q^2)\right]z-(\Lambda(\theta)-q)\left(q+\frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2z)}\right]\neq 0, \qquad (z\in\mathbb{U}^*), \tag{15}$$

where $\Lambda(\theta)$ is given by (10).

Proof. From (14), we have $f\in\mathcal{MK}_{p,q}[b;\mathcal{L}]$ if and only if $-\frac{q^p}{[p]_q}zD_qf\in\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and hence Theorem 2.1 implies

$$z^p\left[-\frac{q^p}{[p]_q}zD_qf(z)*g(z)\right]\neq 0, \qquad (z\in\mathbb{U}^*) \tag{16}$$

where

$$g(z)=\frac{1+[\Lambda(\theta)-q]z}{z^p(1-z)(1-qz)}. \tag{17}$$

We have the identity

$$\left(-\frac{q^p}{[p]_q} zD_q f(z)\right) * g(z) = f(z) * \left(-\frac{q^p}{[p]_q} zD_q g(z)\right) \tag{18}$$

Also from (2)

$$D_q g(z) = \frac{g(z) - g(qz)}{(1-q)z} = \frac{-[p]_q + \left[1 + \Lambda(\theta) - [p]_q(\Lambda(\theta) - q - q^2)\right]z + (\Lambda(\theta) - q)\left(q[p]_q + 1\right)qz^2}{q^p z^{p+1}(1-z)(1-qz)(1-q^2 z)}$$

and therefore

$$-\frac{q^p}{[p]_q} zD_q g(z) = \frac{1 - \left[\frac{1+\Lambda(\theta)}{[p]_q} - \left(\Lambda(\theta) - q - q^2\right)\right]z - (\Lambda(\theta) - q)\left(q + \frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2 z)}. \tag{19}$$

it is simple to check that (15) follows from (16), (18) and (19). Thus, the proof of Theorem 2.2 is completed. □

In particular, from Theorem 2.1 and Theorem 2.2, we obtain the following corollaries for $p = 1$ and $q \to 1$:

Corollary 2.1. *If $f \in \Sigma$, then $f \in \mathcal{MS}^*[b;\mathcal{L}]$ if and only if*

$$z\left[f(z) * \frac{1 + [\Lambda_1(\theta) - 1]z}{z(1-z)^2}\right] \neq 0 \qquad (z \in \mathbb{U}^*), \tag{20}$$

where

$$\Lambda_1(\theta) = \frac{5}{b(4e^{i\theta} + e^{4i\theta})}. \tag{21}$$

Corollary 2.2. *If $f \in \Sigma$, then $f \in \mathcal{MK}[b;\mathcal{L}]$ if and only if*

$$z\left[f(z) * \frac{1 + z + 2(1 - \Lambda_1(\theta))z^2}{z(1-z)^3}\right] \neq 0 \qquad (z \in \mathbb{U}^*), \tag{22}$$

where $\Lambda_1(\theta)$ is given by (21).

Theorem 2.3. *A necessary and sufficient condition for the function f defined by (4) to be in the class $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ is that*

$$1 + \sum_{k=1}^{\infty} \frac{5[k]_q + (4e^{i\theta} + e^{4i\theta})bq^p}{(4e^{i\theta} + e^{4i\theta})bq^p} a_k z^k \neq 0 \qquad (z \in \mathbb{U}^*) \tag{23}$$

Proof. From Theorem 2.1, we find that $f \in \mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ if and only if (9) holds. Since

$$\frac{1}{z^p(1-z)(1-qz)} = \frac{1}{z^p} + (1+q)z^{1-p} + (1+q+q^2)z^{2-p} + (1+q+q^2+q^3)z^{3-p} + \cdots,$$

$(z \in \mathbb{U}^*)$,

hence

$$\frac{1+[\Lambda(\theta)-q]z}{z^p(1-z)(1-qz)} = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(1+\Lambda(\theta)[k]_q\right) z^{k-p},$$

where $\Lambda(\theta)$ is given by (10).

Now a simple computation shows that (9) is identical to (23). Thus, the proof of Theorem 2.3 is completed. □

Theorem 2.4. *A necessary and sufficient condition for the function f defined by (1) to be in the class $\mathcal{MK}_{p,q}[b;\mathcal{L}]$ is that*

$$1+\sum_{k=1}^{\infty} \frac{5[k]_q+(4e^{i\theta}+e^{4i\theta})bq^p}{(4e^{i\theta}+e^{4i\theta})bq^p}\left(1-\frac{[k]_q}{[p]_q}\right) a_k z^k \neq 0 \qquad (z \in \mathbb{U}^*). \tag{24}$$

Proof. From Theorem 2, we find that $f \in \mathcal{MK}_{p,q}[b;\mathcal{L}]$ if and only if (15) holds. Since

$$\frac{1}{z^p(1-z)(1-qz)(1-q^2z)} = \frac{1}{z^p} + (1+q+q^2)z^{1-p} + (1+q+2q^2+q^3+q^4)z^{2-p} + (1+q+2q^2+2q^3+2q^4+q^5+q^6)z^{3-p} + \cdots, \qquad (z \in \mathbb{U}^*).$$

hence

$$\frac{1-\left[\frac{1+\Lambda(\theta)}{[p]_q}-(\Lambda(\theta)-q-q^2)\right]z-(\Lambda(\theta)-q)\left(q+\frac{1}{[p]_q}\right)qz^2}{z^p(1-z)(1-qz)(1-q^2z)} = \frac{1}{z^p} + \sum_{k=1}^{\infty}\left(1+\Lambda(\theta)[k]_q\right)\left(1-\frac{[k]_q}{[p]_q}\right) z^{k-p}, \quad (z \in \mathbb{U}^*).$$

where $\Lambda(\theta)$ is given by (10).

Now a simple computation shows that (24) is identical to (15). Thus, the proof of Theorem 2.4 is completed. □

Theorem 2.5. *If $f \in \Sigma_p$ satisfies the inequality*

$$\sum_{k=1}^{\infty}\left[5[k]_q+3|b|q^p\right]|a_k| < 3|b|q^p \text{ then } f \in \mathcal{MS}^*_{p,q}[b;\mathcal{L}]. \tag{25}$$

Proof. Since

$$\left|1+\sum_{k=1}^{\infty}\frac{5[k]_q+(4e^{i\theta}+e^{4i\theta})bq^p}{(4e^{i\theta}+e^{4i\theta})bq^p}a_k z^k\right|$$
$$\geq 1-\left|\sum_{k=1}^{\infty}\frac{5[k]_q+(4e^{i\theta}+e^{4i\theta})bq^p}{(4e^{i\theta}+e^{4i\theta})bq^p}a_k z^k\right|$$
$$\geq 1-\sum_{k=1}^{\infty}\frac{5[k]_q+3|b|q^p}{3|b|q^p}|a_k|>0.$$

Thus, the inequality (25) holds and our result follows from Theorem 2.3. □

Using similar arguments to those in the proof of Theorem 2.5, we may also prove the next result.

Theorem 2.6. *If $f \in \Sigma_p$ satisfies the inequality*

$$\sum_{k=1}^{\infty}\left[5[k]_q+3|b|q^p\right]\left(1-\frac{[k]_q}{[p]_q}\right)|a_k|<3|b|q^p \tag{26}$$

then $f \in \mathcal{MK}_{p,q}[b;\mathcal{L}]$.

On taking $p=1$ and $q \to 1$, in the above theorems we can easily obtain the similar results for the classes $\mathcal{MS}^*[b;\mathcal{L}]$ and $\mathcal{MK}^*[b;\mathcal{L}]$.

3. Conclusion

In the present paper, two subclasses $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and $\mathcal{MK}_{p,q}[b;\mathcal{L}]$ for meomorphic functions are introduced with the help of q-derivative operator. These classes are associated with the analytic functon $1+\frac{4}{5}z+\frac{z^4}{5}$, which maps the open unit disc $\mathbb{U}=\{z: z\in\mathbb{C}, |z|<1\}$ onto a three-petal leaf-shaped bounded symmetric region. Hadamard products, necessary and sufficient conditions and coefficients estimates are obtained for functions belonging to the classes $\mathcal{MS}^*_{p,q}[b;\mathcal{L}]$ and $\mathcal{MK}_{p,q}[b;\mathcal{L}]$.

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